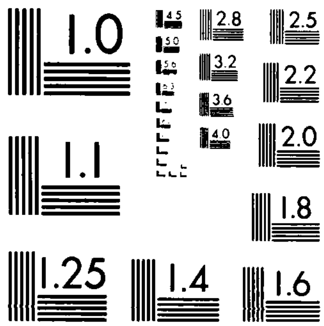
AD-A155 117 ROBUST CONTROL OF MULTIVARIABLE AND LARGE SPACE SYSTEMS (U) HONEYWELL SYSTEMS AND RESEARCH CENTER MINNEAPOLIS MN J D DOYLE ET AL. MAR 85 AFOSR-TR-85-0473 1/2

UNCLASSIFIED F49620-82-C-0090 F/G 12/1 NL



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ROBUST CONTROL OF MULTIVARIABLE AND LARGE SPACE SYSTEMS

J. C. DOYLE

~~DRAFT~~ FINAL REPORT

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85 5 28 279

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 85-0473
6a. NAME OF PERFORMING ORGANIZATION Honeywell, Inc. — 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Systems and Research Center, 2600 Ridgway Parkway, P.O. Box 312, Minneapolis MN 55440	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR — 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F49620-82-C-0090
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A1; WORK UNIT NO.

11. TITLE (Include Security Classification)
ROBUST CONTROL OF MULTIVARIABLE AND LARGE SPACE SYSTEMS, FINAL SCIENTIFIC REPORT, AFOSR /CONTRACT F49620-82-C-0090, 1 JULY 1982 - 30 JUNE 1984

12. PERSONAL AUTHOR(S)
J.D. Doyle and T.B. Cunningham

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Final	FROM 1/7/82 TO 30/6/84	MAR 85	170

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This report, in the form of a set of notes, details Honeywell's research results of the past year in Robust Multivariable Control Theory. These notes are made up of four major parts. Part 0 gives a review of the required notation and mathematical background. Part 1 reviews recent results on the problem of analyzing the performance and robustness properties of systems. Part 2 presents the results on synthesis which are the highlight of this report, and Part 3 outlines how the methods of the previous parts apply to control of large space structures.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. Marc Q. Jacobs	22b. TELEPHONE NUMBER (Include Area Code) (202) 767-4940 — 22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

Honeywell

ROBUST CONTROL OF MULTIVARIABLE AND LARGE SPACE SYSTEMS

J. C. DOYLE

~~DRAFT~~ FINAL REPORT

PREPARED FOR
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
BUILDING 410
BOLLING AIR FORCE BASE, DC 20221
CONTRACT F49620-82-C-0090
1 JULY 1982 - 30 JUNE 1984

SYSTEMS & RESEARCH CENTER
2600 RIDGWAY PARKWAY
P.O. BOX 312
MINNEAPOLIS, MN 55440

PRINTED IN USA

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Part 0. Introduction and Preliminaries

0.0 Introduction and Overview

This report, in the form of a set of notes, details Honeywell's research results of the past year in Robust Multivariable Control Theory. These notes are made up of four major parts. Part 0 gives a review of the required notation and mathematical background. Part 1 reviews recent results on the problem of *analyzing* the performance and robustness properties of systems. Part 2 presents the results on *synthesis* which are the highlight of this report, and Part 3 outlines how the methods of the previous parts apply to control of large space structures.

In the context of these notes, the words analysis and synthesis have specific meanings. **Analysis** is used to describe the process of determining whether a given system has the desired characteristics. In general, this may range from the use of mathematical tools to simulation to experimentation, although analysis is typically applied primarily to describe the former. **Synthesis**, on the other hand, is the process of finding a particular system component to achieve desired characteristics, which are typically expressed in terms of some analysis tools. **Analysis** and **synthesis** are just two aspects of the more general problem of engineering **design**.

The results reported in these notes represent a significant advancement in the state-of-the-art of robust multivariable control design. In addition to providing specific mathematical results, these notes describe a general approach to control problems which is intended to provide the foundations for a new paradigm for control theory broader in scope and content than what has been previously available. This paradigm is introduced in Part 1 on Analysis. An important aspect of this new paradigm is the treatment it gives to model uncertainty.

Modern Control Theory, the dominant paradigm for the past 20 years, has its basis in Stochastic Optimal Control and Estimation Theory [LQG]. This theory essentially restricts model uncertainty to additive noise. The theory provides a methodology for analyzing the impact of noise on system performance and synthesizing to reduce that impact.

The inadequacies of this view of uncertainty became widely accepted in the late 1970's, as robustness to plant uncertainty became a major theme in the Modern Control Theory community [LMC]. Ironically, this involved a renewed interest in the Classical Control paradigm ([Bod],[Hor]), which Modern Control displaced within the theoretical community (if not among practicing engineers). This new direction provided useful design tools, including Singular Value Analysis and Multivariable Loop Shaping ([DSt1],[Doy1],[DSt2]).

While providing an important perspective, as well as practical techniques, the methods based on singular values still require rather restrictive assumptions about uncertainty. In particular, plant uncertainty must essentially be modelled as a single "unstructured perturbation."

The Structured Singular Value (SSV), μ, was developed several years ago to correct this deficiency in singular values ([Doy2],[DWS]). In the context of the general framework discussed in this memo, the SSV provides a very powerful mathematical tool for the analysis of complex systems. Indeed, we believe that this framework together with the SSV and the synthesis techniques discussed later, has the potential to form the basis for a new paradigm for control theory. Part 1 of these notes describes the general framework for control system analysis and synthesis which includes all the viewpoints discussed as special cases. In particular, the assumptions about

uncertainty required by each methodology are compared.

In Part 1 it is shown that each analysis methodology boils down to evaluating

$$\left\|P_{ij}\right\|_{\alpha} \quad \alpha=2,\infty \text{ or } \mu \tag{1}$$

for some transfer function P_{ij}. Thus when the controller is put back into the problem, it involves the synthesis problem

$$\min_{K}\left\|\mathbf{F}_l(P,K)\right\|_{\alpha} \text{ for } \alpha=2,\infty, \text{ or } \mu \tag{2}$$

subject to internal stability of the nominal. Here

$$\mathbf{F}_l(P,K)=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}.$$

The solution of this problem for $\alpha=2$ and ∞ is the focus of Part 2 on Synthesis Theory. The solution presented there unifies the two approaches in a common synthesis framework. The $\alpha = 2$ case was already known and the results are simply a new interpretation. The $\alpha=\infty$ case had been solved only for special cases where P_{12} and P_{21} are square. Also, the existing solutions did not have computational schemes allowing their use on even moderately sized problems. These two limitations, especially the former, restricted the application of the pioneering H_∞ methods to fairly simple problems, such as sensitivity minimization. The new solution presented in Part 2 eliminates these two limitations.

Unfortunately, this new solution for the H_2 and H_∞ suffers from the same limitations imposed by restrictive assumptions about uncertainty as do the underlying analysis methods. While the SSV is a great improvement for analysis, synthesis for the $\alpha=\mu$ case is not yet fully solved. It is shown in Part 1 that μ may be obtained by scaling and applying $\left\|\cdot\right\|_\infty$, so a reasonable approach is to "solve"

$$\min_{K,D}\left\|D\mathbf{F}_l(P,-K)D^{-1}\right\|_\infty \tag{3}$$

by iteratively solving for K and D. With either K or D fixed, the global optimum in the other variable may be found using the μ and H_∞ solutions described in Parts 1 and 2, respectively. Example designs have been done and this scheme seems to work well, but global convergence is not guaranteed. In fact, a counterexample has been constructed where (3) reaches a local minimum which is not global. This is the subject of ongoing research.

As noted above, the main results of these notes are the synthesis solutions of Part 2, particularly for the H_∞ case. This part focuses on the mathematical problem of finding $K \in R_p^{p_2 \times m_2}$ such that

$$\mathbf{F}_l(P,K) \in RH_\infty^{p_1 \times m_1} \tag{4}$$

and

$$\left\|\mathbf{F}_l(P,K)\right\|_\infty \leq \gamma \tag{5}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R_p^{(p_1+p_2)\times(m_1+m_2)}, \quad P_{ij} \in R_p^{p_i \times m_j} \tag{6}$$

and γ are given and

$$\mathbf{F}_l(P,K) = P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}. \tag{7}$$

The important mathematical questions include finding for which γ solutions exist and then parametrizing all solutions to (4) and (5). The notation used here is that R_p denotes proper, real-rational and RH_∞ denotes the subset of R_p analytic in the right-half of the complex plane. The above problem is a natural generalization of the standard (rational) matrix "best approxima-

tion" or "generalized interpolation" problem. The relationship between (4)-(5) and these familiar problems will be discussed in Part 2 of these notes which presents an approach to solving (4)-(5).

A important issue in applying mathematical results of the type developed in these notes to engineering problems is the ability to generate algorithms that compute the solutions in a reliable and efficient manner. These notes pay close attention to this issue by developing the mathematics in a way that makes algorithm development reasonably straightforward. This is often done at the expense of elegance. It's worth noting that virtually all the results in these notes have been implemented in computer software and that this software is currently being used to do experimental designs for non-trivial engineering problems. From a practical engineering point of view, the results have been most encouraging.

The remainder of this part of the notes simply defines notation and reviews some of the specialized methods which are standard within the control theory community but are not typically well-known to mathematicians.

0.1 Definitions and Notation

0.1.0 Notation

SYMBOL	USAGE
$\cdot$	$\dot{x}(t) := \frac{d}{dt}x(t)$
$*$	1. $(x*y)(t) := \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$
	2. A^* = complex-conjugate transpose of complex matrix A
	3. Γ^* = adjoint of operator Γ
1_+, δ	$1_+(t)$ = unit step function; $\delta(t)$ = unit impulse
s	$G(s)$ = two-sided Laplace transform of $g(t)$
$\perp$	orthogonal complement
$\bar{\sigma}$	$\bar{\sigma}(A)$ = largest singular value of matrix A
ρ	$\rho(A)$ = spectral radius of matrix A

$$YX = T\Lambda T^{-1}\ , \qquad \Lambda = \mathrm{diag}(l_1,\ldots l_n)$$

Then columns of T are (possibly nonunique) eigenvectors of YX corresponding to the eigenvalues $\{\lambda_i\}$. It is shown in Lemma 1 at the end of this section that YX has real diagonal Jordan form and that $\Lambda \geq 0$. This is a consequence of $Y \geq 0$ and $X \geq 0$.

Although the eigenvectors are not unique, in the case of a minimal realization they can always be chosen such that

$$\hat{Y} = TYT' = \Sigma,$$
$$\hat{X} = (T^{-1})'XT^{-1} = \Sigma,$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\Sigma^2 = \Lambda$. This new realization will be referred to as a balanced realization (also called internally balanced) [Moo].

Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a balanced realization for G and can be partitioned as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

with corresponding partitioning of the balanced gramian $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$. Suppose $\Sigma_1 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_r)$, $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} \geq \sigma_{r+2} \geq \cdots \geq \sigma_n$. Then it is immediate that the truncated system

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced since

0.3 Gramians and Inner Transfer Functions

0.3.1 Gramians and Balanced Realizations

Suppose $G=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where A is stable. Define the *controllability gramian* Y as

$$Y \triangleq \int_0^\infty e^{At} BB' e^{A't}\, dt$$

and the *observability gramian* as

$$X \triangleq \int_0^\infty e^{A't} C'C e^{At}\, dt.$$

By considering the corresponding matrix differential equations it is easily shown that Y and X satisfy the Lyapunov equations

$$AY + YA' + BB' = 0$$
$$A'X + XA + C'C = 0$$

Note that $Y \geq 0$ and $X \geq 0$. Furthermore, the pair (A,B) is controllable iff $Y>0$ and (C,A) is observable iff $X>0$.

Suppose the state is transformed by nonsingular T to $\hat{x} = Tx$ to yield the realization

$$G = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \quad \text{with} \quad \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}.$$

Then the gramians transform as $\hat{Y} = TYT'$ and $\hat{X} = (T^{-1})'XT^{-1}$. Note that $\hat{Y}\hat{X} = TYXT^{-1}$ so the eigenvalues of the product of the gramians are invariant under state transformation.

Consider the similarity transformation T which gives the eigenvector decomposition

0.2.4 Linear Matrix Equations :

Property 1 : (Solution of Sylvester Equations)

Consider the Sylvester equation

$$AX + XB = C \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$ are given matrices.

Then, there exists a unique solution $X \in \mathbb{R}^{n\times m}$ if and only if $\mathrm{Re}[\lambda_i(A) + \lambda_j(B)] \neq 0, \quad \forall\, i = 1,......,n$ and $j = 1,......,m$.

Remark :

In particular, if $B = A^T$, (1) is called the "Lyapunov Equation" and the necessary and sufficient condition for the existence of unique solution will be that $\mathrm{Re}[\lambda_i(A) + \lambda_j(A)] \neq 0, \ \forall\, i,j = 1,......,n$.

Property 2 : (Solution of Linear Equations)

Consider the linear equation

$$A\,X = B$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are given matrices.

The following statements are equivalent :

(i) there exists a solution $X \in \mathbb{R}^{n\times m}$.

(ii) the columns of $B \in \mathit{Range}(A)$.

(iii) $\mathit{rank}\begin{bmatrix} A & B \end{bmatrix} = \mathit{rank}\begin{bmatrix} A \end{bmatrix}$.

(iv) $\mathrm{Ker}\,(A^T) \subset \mathrm{Ker}\,(B^T)$.

$$\tilde{G}G = ZD.$$

The following lemma characterizes the relationship between zeros of a transfer function and poles of its inverse.

8. Lemma Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with D nonsingular. Then there exists (s_0, x_0) such that

$$(A - BD^{-1}C)x_0 = s_0 x_0 \ , \quad Cx_0 \neq 0$$

iff there exists $u_0 \neq 0$ such that

$$G(s_0)u_0 = 0$$

Proof

(if)

$G(s_0)u_0 = 0$ implies that $G^{-1}(s)$ has a pole at s_0. Thus $\exists (s_0, x_0)$ such that $Cx_0 \neq 0$ and

$$(A - BD^{-1}C)x_0 = s_0 x_0$$

(only if)

Set $u_0 = -D^{-1}Cx_0 \neq 0$. Then

$$G(s_0)u_0 = C(s_0 I - A)^{-1}Bu_0 + Du_0 = Cx_0 - Cx_0 = 0.$$

QED

Proof: The right inverse case will be proven and the left inverse case follows by duality. Suppose $DD^\dagger = I$. Then

$$GG^\dagger = \left[\begin{array}{cc|c} A & BD^\dagger C & BD^\dagger \\ 0 & A-BD^\dagger C & -BD^\dagger \\ \hline C & DD^\dagger C & DD^\dagger \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A & BD^\dagger C & BD^\dagger \\ 0 & A-BD^\dagger C & -BD^\dagger \\ \hline C & C & I \end{array}\right]$$

Conjugating the state by $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ on the left and $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$ on the right yields

$$GG^\dagger = \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & A-BD^\dagger C & -BD^\dagger \\ \hline C & 0 & I \end{array}\right]$$

$$= I$$

Corollary 7 Suppose $D^\dagger$ is a right inverse for D and let

$$\hat{G} = \left[\begin{array}{c|c} A-BD^\dagger C & -BZ \\ \hline D^\dagger C & Z \end{array}\right].$$

Then

$$G\hat{G} = DZ.$$

Corollary 7' Suppose $D^\dagger$ is a left inverse for D and let

$$\hat{G} = \left[\begin{array}{c|c} A-BD^\dagger C & -BD^\dagger \\ \hline ZC & Z \end{array}\right].$$

Then

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]$$

4. Output Injection

$$\dot{x} = Ax + Bu \quad \rightarrow \quad \dot{\hat{x}} = A\hat{x} + Bu + Hy$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A+HC & B+HD \\ \hline C & D \end{array}\right]$$

5. Transpose (Dual)

$$G \rightarrow G^T$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$$

6. Conjugate

$$G \rightarrow G^*$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \rightarrow \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right]$$

7. Inversion

Suppose $D^\dagger$ is a right (left) inverse of D. Then $G^\dagger = \left[\begin{array}{c|c} A-BD^\dagger C & -BD^\dagger \\ \hline D^\dagger C & D^\dagger \end{array}\right]$ is a right (left) inverse of G.

0.2.3 Operations on Linear Systems

1. Cascade

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

$$\begin{aligned} G_1 G_2 &= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right] \end{aligned}$$

Note: This realization may not be minimal.

2. Change of Variables

$$x \to \hat{x} = Tx$$

$$y \to \hat{y} = Ry$$

$$u \to \hat{u} = Pu$$

$$\begin{aligned} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \to \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] &= \begin{bmatrix} T & 0 \\ 0 & R \end{bmatrix} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & P^{-1} \end{bmatrix} \\ &= \left[\begin{array}{c|c} TAT^{-1} & TBP^{-1} \\ \hline RCT^{-1} & RDP^{-1} \end{array}\right] \end{aligned}$$

3. State Feedback

$$u \to \hat{u} + Fx$$

Suppose $G(s)$ is a real-rational transfer matrix which is *proper*, i.e., analytic at $s=\infty$. Then there exists a state-space model (A,B,C,D) such that

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{6}$$

The quadruple (A,B,C,D) is called a *realization* of G. A realization is *minimal* if A has minimal dimension. It is a fact that a realization is minimal if and only if (A,B) is controllable and (C,A) is observable.

A basic object of study will be the transfer function and it will be assumed to have a realization. The next section describes standard operations on linear systems in terms of transfer functions and their realizations.

0.2.2 Transfer Functions

Consider the linear, time-invariant, ordinary differential equation described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the output. The A, B, C, and D are appropriately dimensioned real matrices.

Associated with (1) is the convolution equation

$$\begin{aligned} y(t) &= (g * u)(t) \\ g(t) &= Ce^{At}B\,1_+(t) + D\delta(t) \end{aligned} \tag{2}$$

and, upon taking Laplace transforms, the resulting transfer function is

$$\begin{aligned} y(s) &= G(s)u(s) \\ G(s) &= C(sI-A)^{-1}B + D \end{aligned} \tag{3}$$

To expedite calculations involving transfer functions the notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(sI-A)^{-1}B + D \tag{4}$$

will be adopted. Note that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a real block matrix, not a transfer function. The product of two transfer functions is, of course, the cascade of the two systems or just the multiplication of two rational matrices. The convention will be adopted that the product of a matrix and a transfer function is a transfer function defined as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq \left[\begin{array}{c|c} X_{11}A + X_{12}C & X_{11}B + X_{12}D \\ \hline X_{21}A + X_{22}C & X_{21}B + X_{22}D \end{array}\right]. \tag{5}$$

A similar convention holds for right multiplication by a matrix.

$$\dot{x} = Ax \;, \quad x(0) = x_0$$
$$y = Cx.$$

The system, or the pair (C,A), is *observable* if, for every $t_1>0$, the function $y(t)$, $t \in [0,t_1]$, uniquely determines the initial state x_0.

Theorem 1':

The following are equivalent:

(i) (C,A) is observable.

(ii) The matrix $\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$ has independent columns.

(iii) The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has independent columns for all λ in $\mathbb{C}$.

(iv) The eigenvalues of $A+HC$ can be freely assigned by suitable choice of H.

(v) (A',C') is controllable.

The system, or the pair (C,A), is ***detectable*** if $A+HC$ is stable for some H.

Theorem 2':

The following are equivalent:

(i) (C,A) is detectable

(ii) The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has independent columns for all Re $\lambda \geq 0$.

(iii) (A',C') is stabilizable.

0.2 Linear Systems

0.2.1 Controllability and Observability

Consider the system

$$\dot{x} = Ax + Bu \ , \quad x(0) = 0. \tag{1}$$

The system or the pair (A,B) is *controllable* if, for each time $t_1 > 0$ and final state x_1, there exists a (continuous) input $u(\cdot)$ such that the solution of (1) satisfies $x(t_1) = x_1$.

Theorem 1

The following are equivalent:

(i) (A,B) is controllable.

(ii) The matrix $\left[B, AB, A^2B, \ldots\right]$ has independent rows.

(iii) The matrix $\left[A-\lambda I, B\right]$ has independent rows for all λ in $\mathbb{C}$

(iv) The eigenvalues of $A+BF$ can be freely assigned by suitable choice of F.

The matrix A is said to be *stable* if all its eigenvalues satisfy $\mathrm{Re}\lambda < 0$. The system, or the pair (A,B), is *stabilizable* if there exists an F such that $A+BF$ is stable.

Theorem 2

The following are equivalent:

(i) (A,B) is stabilizable.

(ii) The matrix $\left[A-\lambda I, B\right]$ has independent rows for all $\mathrm{Re}\lambda \geq 0$.

We will now consider the dual notions of observability and detectability with the system

$$\|G\|_\infty = \sup\left\{\|Gf\|_2 : f \in H_2(j\mathbb{R}, \mathbb{C}^n),\ \|f\|_2 \leq 1\right\}.$$

$L_\infty(j\mathbb{R},\mathbb{C}^{m\times n})$: Banach space of (essentially) bounded matrix-valued functions, with norm

$$\|F\|_\infty := \operatorname{ess\,sup}_{\omega} \bar{\sigma}\left[F(j\omega)\right].$$

$H_\infty(j\mathbb{R},\mathbb{C}^{m\times n})$: subspace of functions $F(s)$ analytic and bounded in Re $s>0$.

P_{H_2}, $P_{H_2^\perp}$: the orthogonal projections from $L_2(j\mathbb{R},\mathbb{C}^{m\times n})$ onto $H_2(j\mathbb{R},\mathbb{C}^{m\times n})$, $H_2(j\mathbb{R},\mathbb{C}^{m\times n})^\perp$ respectively.

Prefix R denotes real-rational and the prefix B denotes the unit ball. The symbol $R_p^{(m\times n)}$ denotes proper real-rational matrices. Sometimes the spaces are abbreviated as $L_2(\mathbb{R})$, etc. or as L_2, etc. when context determines the arguments.

The Fourier transform yields the following [isometric] isomorphisms:

$$L_2(\mathbb{R},\ \mathbb{C}^{m\times n}) \cong L_2(j\mathbb{R},\ \mathbb{C}^{m\times n})$$
$$H_2(\mathbb{R},\ \mathbb{C}^{m\times n}) \cong H_2(j\mathbb{R},\ \mathbb{C}^{m\times n})$$
$$H_2(\mathbb{R},\ \mathbb{C}^{m\times n})^\perp \cong H_2(j\mathbb{R},\ \mathbb{C}^{m\times n})^\perp$$

The norms on these spaces are all denoted by $\|\bullet\|_2$.

A useful fact is that the norm of a matrix G in $L_\infty(j\mathbb{R},\ \mathbb{C}^{m\times n})$ equals the norm of the corresponding multiplication operator

$$f \to Gf\ :\ L_2(j\mathbb{R},\ \mathbb{C}^n) \to L_2(j\mathbb{R},\ \mathbb{C}^m);$$

that is,

$$\|G\|_\infty = \sup\left\{\|Gf\|_2 : f \in L_2(j\mathbb{R},\ \mathbb{C}^n),\ \|f\|_2 \le 1\right\}.$$

It also equals the norm of the operator restricted to $H_2(j\mathbb{R},\ \mathbb{C}^n)$:

0.1.1 Function Spaces

Continuous time domain

$L_2(\mathbb{R},\mathbb{C}^{m\times n})$: Hilbert space of matrix-valued functions on $\mathbb{R}$, with inner product

$$<f,g> := \int_{-\infty}^{\infty} \mathrm{trace}\left[f(t)^* g(t)\right] dt.$$

$H_2(\mathbb{R},\mathbb{C}^{m\times n})$: subspace of functions zero for $t<0$.

$H_2(\mathbb{R},\mathbb{C}^{m\times n})^{\perp}$: subspace of functions zero for $t>0$.

P_{H_2} and $P_{H_2^{\perp}}$: the orthogonal projections from $L_2(\mathbb{R},\mathbb{C}^{m\times n})$ onto $H_2(\mathbb{R},\mathbb{C}^{m\times n})$, $H_2(\mathbb{R},\mathbb{C}^{m\times n})^{\perp}$ respectively.

Continuous frequency domain

$j\mathbb{R}$: imaginary axis.

$L_2(j\mathbb{R},\mathbb{C}^{m\times n})$: Hilbert space of matrix-valued functions on $j\mathbb{R}$, with inner product

$$<F,G> := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\left[F(j\omega)^* G(j\omega)\right] d\omega.$$

$H_2(j\mathbb{R},\mathbb{C}^{m\times n})$: subspace of functions $F(s)$ analytic in $\mathrm{Re}\, s>0$ and satisfying

$$\sup_{\sigma>0}\int_{-\infty}^{\infty} \mathrm{trace}\left[F(\sigma+j\omega)^* F(\sigma+j\omega)\right] d\omega < \infty$$

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}' + B_1 B_1' = 0$$

$$A_{11}'\Sigma_1 + \Sigma_1 A_{11} + C_1' C_1 = 0.$$

It can also be shown (Silverman and Pernebo) that a *minimal* realization for G_r is stable, although in certain (non-generic) cases A_{11} may have uncontrollable or unobservable $j\omega$-axis eigenvalues.

Lemma 1 Product of Positive Semi-Definite Matrices is Similar to a Positive Semi-Definite Matrix

Proof: Let X and Y be positive semi-definite. First perform an orthogonal transformation so that

$$X \rightarrow \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Lambda_1 > 0 \text{ diagonal}, \quad Y \rightarrow \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}' & Y_{22} \end{bmatrix} \geq 0$$

By this transformation XY is similar to $\begin{bmatrix} \Lambda_1 Y_{11} & \Lambda_1 Y_{12} \\ 0 & 0 \end{bmatrix}$. Now

$$\begin{bmatrix} \Lambda_1 Y_{11} & \Lambda_1 Y_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Lambda_1^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda_1^{1/2} Y_{11} \Lambda_1^{1/2} & \Lambda_1^{1/2} Y_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1^{-1/2} & 0 \\ 0 & I \end{bmatrix},$$

and it is easy to find a matrix Z such that

$$\begin{bmatrix} \Lambda_1^{1/2} Y_{11} \Lambda_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda_1^{1/2} Y_{11} \Lambda_1^{1/2} & \Lambda_1^{1/2} Y_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix}$$

(Z exists because the columns of $\Lambda_1^{1/2} Y_{11}$ span the columns of $\Lambda_1^{1/2} Y_{12}$ owing to the fact that Y is positive semi-definite). The left hand side of this last equation is positive semi-definite and similar to XY. If $XY \neq 0$ it is possible to find a matrix T_1 such that

$$T_1^{-1}ABT_1 = \Lambda = \mathrm{diag}\left\{(\lambda_1, \ldots, \lambda_K, 0, \ldots, 0)\right\}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$.

Q.E.D.

Now consider two gramians X and Y. Let us suppose $XY \neq 0$, so that T_1 can be chosen as above:

$$T_1^{-1}XYT_1 = \Lambda = \mathrm{diag}\left\{(\lambda_1, \ldots, \lambda_K, 0, \ldots, 0)\right\}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$. Under this transformation the gramians become

$$Q = T_1^{-1}X(T_1^{-1})', \quad R = T_1'YT_1,$$

and $QR = \Lambda$. Because Q and R are symmetric, $RQ = \Lambda' = \Lambda = QR$ and so Q, R and Λ commute. Both Q and R must leave the eigenspaces of Λ invariant and so are of the form

$$Q = \mathrm{diag}\left\{Q_1, \ldots, Q_l, E\right\}$$

$$R = \mathrm{diag}\left\{\lambda_{j_1}Q_1^{-1}, \ldots, \lambda_{j_l}Q_l^{-1}, F\right\}$$

where Q_i is a square matrix whose size equals the dimension of the λ_{j_i} eigenspace of Λ and $EF = 0$ where E and F are square matrices the size of the kernel of Λ. Of course, all the Q_i's are summetric so it is possible to find an orthogonal matrix

$$W = \mathrm{diag}\left\{W_1, \ldots, W_l, W_{l+1}\right\}$$

such that $W^{-1}Q(W^{-1})'$ and $W_{l+1}'FW_{l+1}$ are diagonal. Note that this same W gives a diagonal $W'RW$ and leaves Λ alone.

The transformation $T_2 = T_1 W$ diagonalizes both gramians. It is now obvious how to construct T_3 so that $T_3^{-1} X (T_3^{-1})'$ and $T_3' Y T_3$ are diagonal and the controllable and observable portions are equal.

0.3.2 Inner Transfer Functions

Let $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then G is inner if $G^*G=I$ and co-inner if $GG^*=I$. Note that G need not be square. Inner and co-inner are dual notions and are often called all-pass.

If $G \in R_p^{p\times m}$, $p>m$ is inner then any $G_\perp \in R_p^{(p-m)\times m}$ is called a complementary inner factor (CIF) if $\begin{bmatrix} G & G_\perp \end{bmatrix}$ is square and inner. The dual notion of complementary co-inner factor is defined in the obvious way.

The following lemma is useful in characterizing inner transfer functions in terms of a realization.

Lemma 1. Suppose $\exists\ X=X' \in \mathbb{R}^{n\times n}$ such that

i) $A'X + XA + C'C = 0$

ii) $B'X + D'C = 0$

Then $G^*G = D'D$.

Proof: Suppose that i) and ii) hold. Then conjugating the state of

$$G^*G = \left[\begin{array}{cc|c} A & 0 & B \\ -C'C & -A' & -C'D \\ \hline D'C & B' & D'D \end{array}\right]$$

by $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ on the left and $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ on the right yields

$$G^*G = \left[\begin{array}{cc|c} A & 0 & B \\ -(A'X+XA+C'C) & -A' & -(XB+C'D) \\ \hline B'X+D'C & B' & D'D \end{array}\right]. \tag{1}$$

Now, applying i) and ii) yields

$$G^*G = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A' & 0 \\ \hline 0 & B' & D'D \end{array}\right]$$

$= D'D$

By duality, we have the following

Lemma 1' Suppose $\exists\, Y = Y' \in \mathbb{R}^{n \times n}$ such that

i) $AY + YA^T + BB' = 0$

ii) $CY + DB' = 0$

Then $GG^* = DD'$.

These two lemmas immediately lead to one characterization of inner matrices in terms of their state space representation. Simply add the condition that $D'D=I$ $(DD'=I)$ to lemma 1 (1') to get $G^*G=I$ $(GG^*=I)$. Furthermore, by adding a few additional assumptions, the conditions in the lemmas become necessary as well as sufficient. This leads to the following complete characterization of stable inner transfer functions in terms of a minimal realization.

Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is stable and minimal. Then the gramians X and Y satisfying

$$A'X + XA + C'C = 0 \tag{2}$$

$$AY + YA' + BB' = 0 \tag{3}$$

exist and are unique.

Corollary 1 G is inner iff

i) $B'X + D'C = 0$

ii) $D'D = I$

Corollary 1' G is co-inner iff

i) $CY + DB' = 0$

ii) $DD' = I$

Proof Sufficiency of i) and ii) follows immediately from the lemmas. For necessity, suppose $G^*G=I$. From 1) and 2) this implies that

$$\left[\begin{array}{c|c} A & B \\ \hline B'X+D'C & 0 \end{array}\right] = 0. \tag{4}$$

$$D'D = I \tag{5}$$

Since (A,B) is controllable, (4) implies that $B'X+D'C=0$. The co-inner case follows by duality.

This characterization of inner transfer functions is from [AnV] and plays an important role in the synthesis theory. It allows the construction of inner transfer functions by solving algebraic equations.

0.4 Linear Fractional Transformations

Suppose $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R_p^{(p_1+p_2)\times(m_1+m_2)}$, $\Delta \in R_p$, $K \in R_p^{m_2 \times p_2}$. We will adopt the notation

$$\mathbf{F}_l(P,K) \triangleq P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21} \tag{1}$$

and

$$\mathbf{F}_u(P,\Delta) \triangleq P_{22} + P_{21}\Delta(I-P_{11}\Delta)^{-1}P_{12} \tag{2}$$

The linear fractional transformations (LFT) are illustrated in Figure 1. The l denotes that the second argument is fed back in the *lower* block, and the u denotes feedback in the upper block.

An important property of LFT's is that any interconnection of LFT's is again an LFT. Suppose $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$. Then

$$\mathbf{F}_l(P,\mathbf{F}_l(J,Q)) = \mathbf{F}_l(T,Q) \tag{3}$$

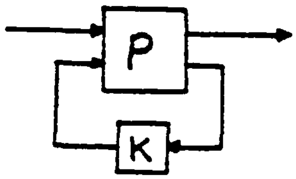
$$\mathbf{F}_u(J,\mathbf{F}_u(P,\Delta)) = \mathbf{F}_u(T,\Delta) \tag{4}$$

where

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} P_{11}+P_{12}J_{11}(I-P_{22}J_{11})^{-1}P_{21} & P_{12}(I-J_{11}P_{22})^{-1}J_{12} \\ J_{21}(I-P_{22}J_{11})^{-1}P_{21} & J_{22}+J_{21}P_{22}(I-J_{11}P_{22})^{-1}J_{12} \end{bmatrix} \end{aligned} \tag{5}$$

Equations (3) and (4) are illustrated in Figure 2. Note that if $\mathbf{F}_l(J,Q)$ is a parametrization of a controller, $\mathbf{F}_l(T,Q)$ is affine if and only if $T_{22}=0$. This type of controller parametrization will play an important role in the synthesis theory.

Figure 1.



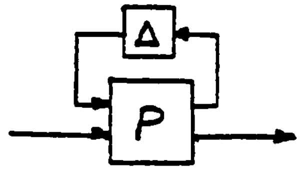
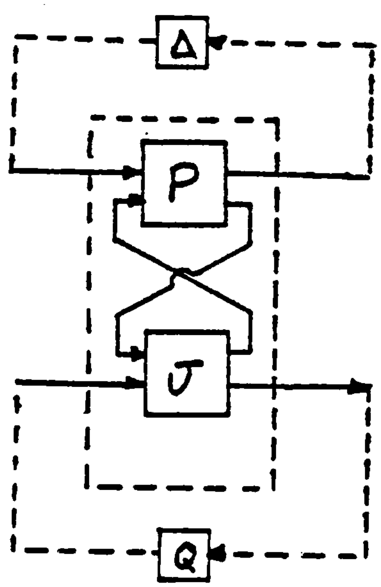


Figure 2.



Part 1. Analysis

1.1 Introduction

This part of the notes briefly reviews recent results on the problem of analyzing the performance and robustness properties of systems ([Doy2],[DWS]). The main goal is to motivate from a control theory point of view the synthesis problems considered in part 2 of these notes. A secondary goal is to describe a general approach to control problems which is intended to provide the foundation for a new paradigm for control theory broader in scope and content than that provided by Classical or Modern Control Theory. An important aspect of this new paradigm is the treatment it gives to model uncertainty.

Modern Control Theory, the dominant paradigm for the past 20 years, has its basis in Stochastic Optimal Control and Estimation Theory [LQG]. This theory essentially restricts model uncertainty to additive noise. The theory provides a methodology for analyzing the impact of noise on system performance and synthesizing to reduce that impact.

The inadequacies of this view of uncertainty became widely accepted in the late 1970's, as robustness to plant uncertainty became a major theme in the Modern Control Theory community [LMC]. Ironically, this involved a renewed interest in the Classical Control paradigm ([Bod],[Hor]), which Modern Control displaced within the theoretical community (if not among practicing engineers). This new direction provided useful design tools, including Singular Value Analysis and Multivariable Loop Shaping ([DSt1],[Doy1],[DSt2]).

While providing an important perspective, as well as practical techniques, the methods based on singular values still require rather restrictive assumptions about uncertainty. In particular, plant uncertainty must

essentially be modelled as a single "unstructured perturbation."

The Structured Singular Value (SSV), μ, was developed several years ago to correct this deficiency in singular values ([Doy2],[DWS]). In the context of the general framework discussed in this memo, the SSV provides a very powerful mathematical tool for the analysis of complex systems. Indeed, we believe that this framework together with the SSV and the synthesis techniques discussed later, has the potential to form the basis for a new paradigm for control theory.

The remainder of this part of the notes describes the general framework for control system analysis and synthesis which includes all the viewpoints discussed as special cases. In particular, the assumptions about uncertainty required by each methodology are compared. In this context, the words analysis and synthesis have specific meanings.

Analysis is used to describe the process of determining whether a given system has the desired characteristics. In general, this may range from the use of mathematical tools to simulation to experimentation, although analysis is typically applied primarily to describe the former. **Synthesis**, on the other hand, is the process of finding a particular system component to achieve desired characteristics, which are typically expressed in terms of some analysis tools. **Analysis** and **synthesis** are just two aspects of the more general problem of engineering **design**.

The discussion which follows first considers analysis, then briefly touches on synthesis. The next part on Synthesis Theory will take up that question in more detail.

1.2 Analysis Framework and Background

1.2.1 General Framework

Various modelling assumptions will be considered and the impact of these assumptions on analysis and synthesis methods will be explored. Consider the diagram in Figure 1. This is the general framework to be considered. Models of this form are typically constructed from components which also have this form. The nominal model provides the basic interconnection structure between the signals, perturbations and controller, as shown. It has three inputs and outputs, each consisting of a vector of signals.

As typical examples, consider the following filtering and control problems. First, a simple filtering problem is given in the diagram in Figure 2. This may be rearranged as shown in Figure 3 to fit the general framework. In order to simplify the diagram, no perturbation was included. A typical control problem might look like the diagram in Figure 4 where again, for simplicity, no perturbations are included. This too can be rearranged to fit the general framework, although the diagram is complicated.

Any system may be rearranged to fit the form of this general framework. Although the interconnection structure can become quite complicated for complex systems, many software packages are available which could be used to generate the interconnection structure from system components.

Note that uncertainty may be modelled in two ways, either as external inputs or as perturbations to the nominal model. The performance of a system is measured in terms of the behavior of the outputs or errors. The assumptions which characterize the uncertainty, performance and nominal model determine the analysis techniques which must be used.

The most fundamental assumption that is made throughout is that the nominal model is a finite dimensional ordinary differential equation and is linear and time invariant (LTIODE). The uncertain inputs are assumed to be either filtered white noise or weighted L_p signals. Performance is measured as weighted output variance or weighted output L_p norm. The perturbations are assumed to be themselves LTIODE's which are norm-bounded as input-output operators. Various combinations of these assumptions form the basis for all the standard linear systems analysis tools.

Given that the nominal model is an LTIODE, the interconnection system has the form

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right] \tag{1}$$

and the total system is a linear fractional transformation on the perturbation and the controller given by

$$\begin{aligned} e &= \mathbf{F}_u(\mathbf{F}_l(P,K),\Delta)\, u \\ &= \mathbf{F}_l(\mathbf{F}_u(P,\Delta),K)\, u \end{aligned} \tag{2}$$

Since the focus of the current discussion is on analysis methods, the controller may be viewed as just another system component and absorbed into the interconnection structure. Thus the analysis framework reduces to the diagram in Figure 5 where

$$\begin{aligned} e &= \mathbf{F}_u(P,\Delta)\, u \\ &= \left[P_{22} + P_{21}\Delta(I-P_{11}\Delta)^{-1}P_{12}\right] u \end{aligned} \tag{3}$$

Note that the P's in (2) and (3) are not necessarily the same. Table 1 and the discussion which follows summarize the various assumptions and resulting analysis and synthesis tools. In each case, stability of the nominal must be evaluated. Since P is assumed to have the state-space representation

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad . \tag{4}$$

this may be done by checking that all eigenvalues of A lie in the open lhp. There are alternatives to this approach but, for simplicity, it will be assumed that the nominal plant, with controller is closed loop stable in the sense that all eigenvalues of A are in the open lhp.

Given nominal stability, the entries in the table may be interpreted as filling in the following general performance/robustness theorem:

General Analysis Theorem (GAT)

Given

Input Assumptions

and

Perturbation Assumptions

Then

Performance Specification

if and only if

Analysis Test.

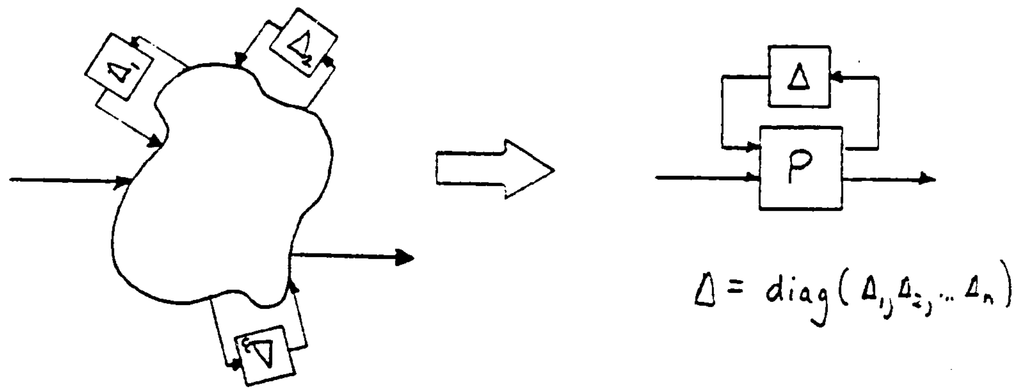


Figure 1

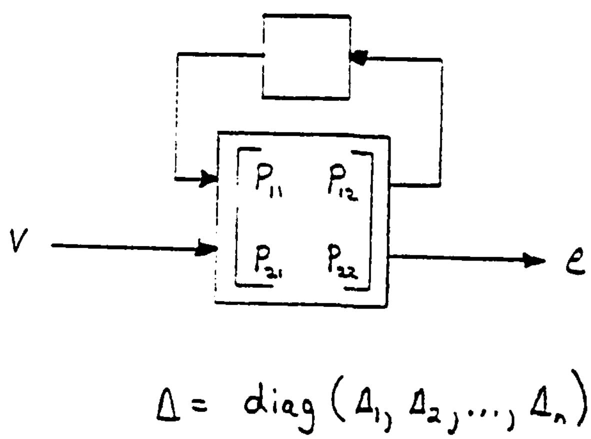


Figure 2

Consider the system in Figure 2. Stability and performance analysis of this system requires a new matrix function, the structured singular value (SSV), denoted by μ. Before proceeding with Case 3, a digression to discuss μ will be taken. For details, see [Doy2].

1.3 Structured Singular Values

1.3.1 Introduction

This chapter considers the problem of stability with structured uncertainty and of performance in the presence of structured uncertainty. Typically, uncertainty is present throughout a system. Suppose that a system is built from components which are themselves uncertain and that component uncertainty is modelled as norm-bounded perturbations. This situaton can be rearranged to fit the general framework but the perturbation for the total system has structure. This can be seen schematically in Figure 1.

Note that the interconnection model P can always be chosen so that Δ is block diagonal, and by absorbing any weights, $\|\Delta\|_\infty < 1$. The results of Case 2b can be applied in two ways:

1) $\|P_{11}\|_\infty \le 1$ implies stability, but *not* conversely. This can be arbitrarily conservative, in that stable systems can have arbitrarily large $\|P_{11}\|_\infty$.

2) Test for each Δ_i individually. This can be arbitrarily optimistic because it ignores interaction between the Δ_i.

The difference between the bounds obtained in 1) and 2) can be arbitrarily far apart. Only when they are close can conclusions be made about the general case with structured uncertainty.

These two limitations of Case 2 (and 1) have motivated much of the research described in these notes. The result is a new paradigm described in Case 3. The problem in Case 3 involves exactly that of structured uncertainty.

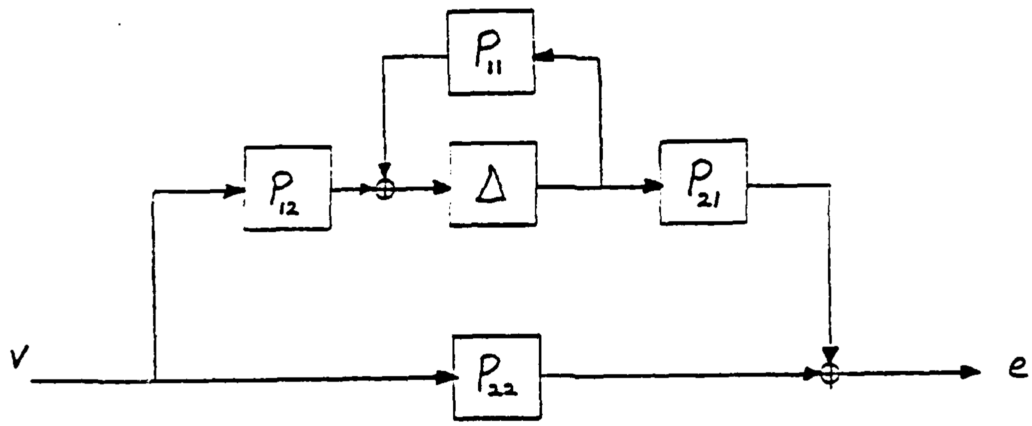


Figure 1.

inputs when $\Delta = 0$ but response when $\Delta \neq 0$ is not known. Only crude bounds can be obtained with the methods of Case 2.

An additional limitation of Case 2b is that all plant uncertainty must be modelled as a single norm-bounded perturbation. Typically, uncertainty is present throughout a system. Suppose that a system is built from components which are themselves uncertain and that component uncertainty is modelled as norm-bounded perturbations. This situation can be rearranged to fit the general framework but the perturbation for the total system has structure. The problem of structured uncertainty is taken up in the next chapter.

feedback problems where both types of uncertainty have significant impacts on system performance. Case 2 has attracted a great deal of research interest recently, and is currently a popular new paradigm in the multivariable control community. Although implicit in the methods of classical control ([Bod],[Hor]) and some modern work (e.g. Zames' conic sector theory [Zam1] and, more generally, the input-output stability theory of the 1960's [DeV]), the approach did not gain wide attention until the late '70's.

The current interpretation is a consequence of research done in the late '70's. ([Doy1],[DSt2],[LMC]). This interpretation involves singular values as an analysis method and singular value loop shaping as a synthesis approach. The so-called LQG Loop Transfer Recovery (LQG/LTR) combines the synthesis methods of Case 1 with the analysis methods of Case 2 to produce a hybrid synthesis method. This gives an ad hoc approach to Case 2 that can be effective for many multivariable problems.

Another approach to synthesis for Case 2 is the so called H_∞ or L_∞/H_∞ methods introduced to the control community by Zames [Zam2] and Helton [Hel], and developed further by many others (eg. [FHZ]). The L_∞/H_∞ methods for Case 2 are analogous to the L_2/H_2 methods of Case 1 with the exception that for Case 2 the L_∞ rather than L_2 norm is optimized. The solution to the general L_∞/H_∞ problem will be presented in the Synthesis part of these notes.

The main objection to Case 2 is the restrictive assumptions about uncertainty (recall this was also the objection to Case 1). Although case 2 allows both uncertain inputs and perturbations, analysis can be performed for *either* individually *but not both* together. Thus a system can be shown to remain stable when perturbed and have acceptable response to uncertain

some additional technical difficulties and is not the focus of these notes.

The following theorem treats internal stability

Theorem 2 $\mathbf{F}_u{}'(P,\Delta)$ is internally stable for all $\Delta \in BRH_\infty$
iff $\|P_{11}\|_\infty \leq 1$

Note that input-output stability of $\mathbf{F}_u(P,\Delta)$ is not necessarily the same as internal stability. In particular, the following statement is not true:

Not–A–Theorem $\|e\|_2 < \infty$ for all $\|u\|_2 \leq 1$ and $\Delta \in BRH_\infty$
iff $\|P_{11}\|_\infty \leq 1$

Counterexample Suppose $\|P_{11}\|_\infty > 1$ but $P_{21} \equiv 0$.

From now on stability will mean *internal* stability, but be denoted by $\|e\|_2 < \infty$ in the table, even though this is definitely an abuse of notation. Note that generically this distinction between internal and i-o stability does not exist.

As in Case 1, it is essential to allow weights on inputs, outputs and perturbations. As before, these weights may be absorbed into the nominal model. This allows, without the loss of generality, the use of signals and perturbations which are in unweighted unit balls. Thus implementation of the analysis tools requires only a method for constructing interconnected systems and a method for evaluating the appropriate norm. The former applies to all cases, whereas the latter requires a different norm in each case.

Note that in Case 2 both uncertain inputs and uncertain plants can be handled with the same analysis tool. This approach is particularly useful for

1.2.3 L_∞ Frequency Domain Methods

Case 2 involves an attempt to correct some of the deficiencies of Case 1 by moving to an unknown but bounded (in an L_2 sense) framework. This allows both types of plant uncertainty to be handled in a common framework, albeit in a limited manner.

Case 2a is an L_2 version of Case 1a. The input is constrained to lie in BL_2 as a time signal (unit ball in L_2) and the performance is specified in terms of the output's L_2 norm. With no perturbation, the analysis test involves simply the L_2 induced operator norm, i.e. L_∞ on the transfer function P_{22}.

The *GAT* in this case is

Theorem 1 $\quad \|e\|_2 \le 1$ for all $\|u\|_2 \le 1$

iff $\quad \|P_{22}\|_\infty \le 1$

Although this theorem is a trivial restatement of the definition of induced norm, it means that the analysis test is an exact characterization of the performance requirement.

Case 2b is significant departure from the previous three. It involves maintenance of stability in the presence of perturbations. The block diagram for $F_u'(P,\Delta)$ is shown in Figure 1. There are many ways to state the *GAT* for this case, depending on the desired notion of stability and assumptions on Δ. The distinctions are somewhat subtle, but are important from a theoretical point of view. Nevertheless, they do not significantly impact the application of the theory.

The Δ's are assumed to be LTIODE's, so that $\Delta \in RH_\infty$. The assumptions $\Delta \in \mathbb{C}$ or $\Delta \in \mathbb{C}H_\infty$ give the same result. The distributed case, $\Delta \in H_\infty$, causes

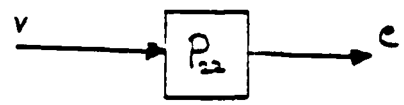


Figure 1

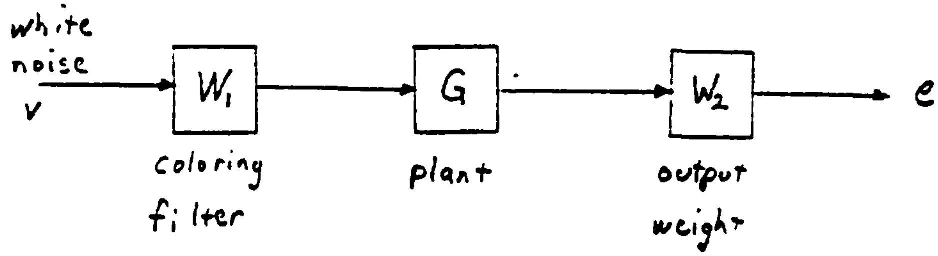


Figure 2

larly critical in feedback systems.

1.2.2. Stochastic

Case 1a involves unit covariance white noise input with output variance as the evaluation criteria. Since no perturbation is allowed, the problem reduces to the diagram in Figure 1 and $E(e^Te) = \left\|P_{22}\right\|_2^2$. Note that colored noise or weighted variance could be used as shown in Figure 2. This reduces to the general case by absorbing the weights W_1 and W_2 into P_{22} as $P_{22} = W_2GW_1$. In practice, it is essential to use weights to reflect spatial and frequency variations in inputs, perturbations and output specifications, but in every case, these weights may be absorbed into nominal model.

In Case 1b the input is an uncertain delta function, which is equivalent to uncertain initial conditions. The performance specification is the expected value of the L_2-norm of the output.

Case 1 forms the foundation of Stochastic Optimal Control Theory. Case 1a includes the standard linear stochastic filtering problem and Case 1b includes the standard linear quadratic optimal control problem. These are combined to obtain the full LQG problem, which is again Case 1a. These assumptions and resulting analysis methods have been the dominant paradigm in the control community for over 20 years.

The development of this paradigm has stimulated extensive research efforts and been responsible for important technological innovation, particularly in the area of estimation. The theoretical contributions include a deeper understanding of linear systems and improved computational methods for complex systems through state-space techniques. The major limitation of this theory is the lack of formal treatment of uncertainty in the plant itself. By allowing only additive noise for uncertainty, the stochastic theory ignored this important practical issue. Plant uncertainty is particu-

Case	Input Assumptions	Performance Specifications	Perturbation Assumption
1a	$E(u(t)u(\tau)^T) = I\delta(t-\tau)$	$E(e(t)^T e(t)) \leq 1$	$\Delta = 0$
1b	$u = u_o\delta(t)$ $E(u_o u_o^T) = I$	$E(\|\|e\|\|_2^2) \leq 1$	
2a	$\|\|u\|\|_2 \leq 1$	$\|\|e\|\|_2 \leq 1$	$\Delta = 0$
2b	$\|\|u\|\|_2 \leq 1$	$\|\|e\|\|_2 < \infty$	$\|\|\Delta\|\|_\infty < 1$
3a	$\|\|u\|\|_2 \leq 1$	$\|\|e\|\|_2 < \infty$	$\|\|\Delta\|\|_\infty < 1$ $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$
3b	$\|\|u\|\|_2 \leq 1$	$\|\|e\|\|_2 \leq 1$	

Case	Analysis Test	Synthesis Method	Comments
1a	$\frac{1}{2\pi}\|\|P_{22}\|\|_2 \leq 1$	Stochastic Optimal Control, (LQG), Wiener-Hopf	Dominant paradigm in 60's and 70's. Does not allow perturbations in analysis or synthesis.
1b			
2a	$\|\|P_{22}\|\|_\infty \leq 1$	Singular Value Loop Shaping, H_∞ Optimization	Recent renewed interest including new techniques for optimal control with H_∞ requirements.
2b	$\|\|P_{11}\|\|_\infty \leq 1$		
3a	$\|\|P_{11}\|\|_\mu \leq 1$	μ-synthesis	Provides both robust stability and robust performance with structured perturbations.-
3b	$\|\|P\|\|_\mu \leq 1$		

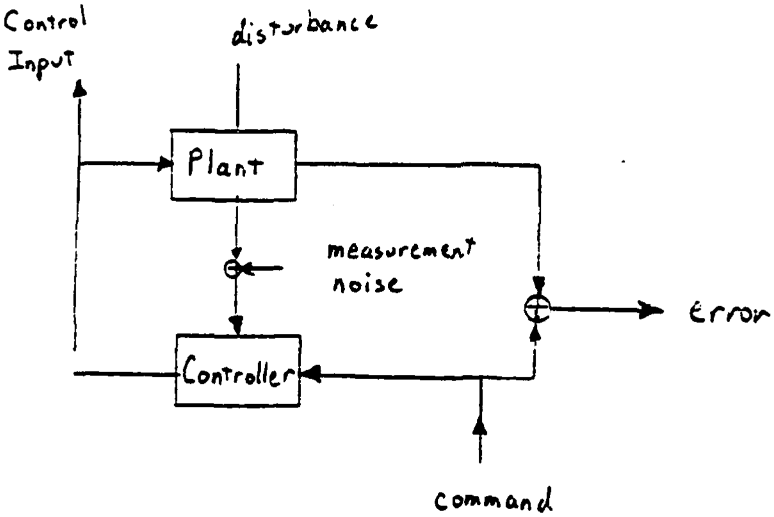
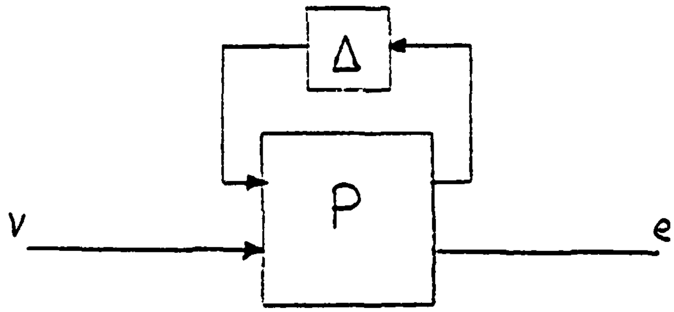


Figure 4.



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Figure 5.

The details of each case will be considered in the following sections.

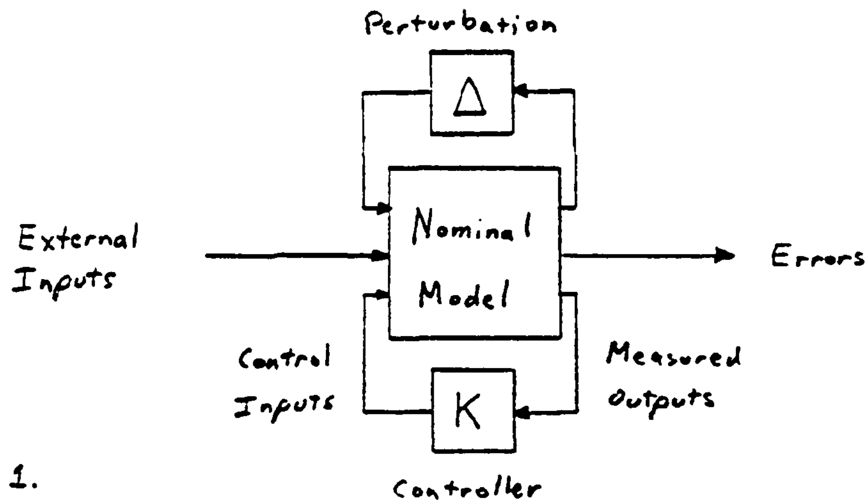


Figure 1.

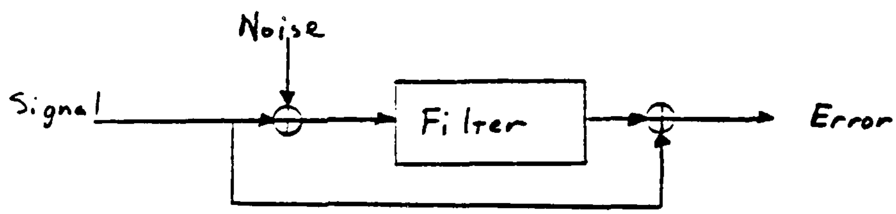


Figure 2.

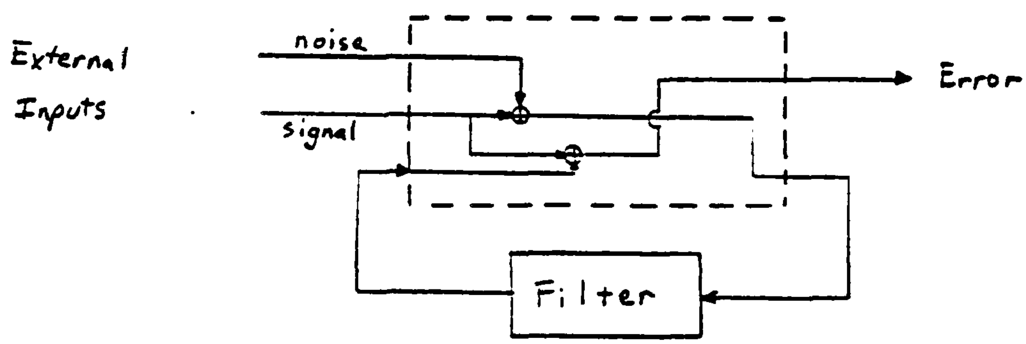


Figure 3.

1.3.2 SSV for Constant Matrices

The problem is to test for $det(I - M\Delta) \neq 0$ for *sets* of Δ. Two standard results are

1) $det(I - M\Delta) \neq 0 \quad \forall\ \Delta \in \left\{ \Delta \,\middle|\, \bar{\sigma}(\Delta) < 1 \right\}$

iff $\quad \bar{\sigma}(M) \leq 1$

2) $det(I - M\Delta) \neq 0 \quad \forall\ \Delta \in \left\{ \lambda I \,\middle|\, \lambda \in \mathbb{C},\ |\lambda| < 1 \right\}$

iff $\quad \rho(M) \leq 1 \quad$ where $\rho(M) = \max_i |\lambda_i(M)|$

As a generalization, consider a function μ with the properties that $\mu(\alpha M) = |\alpha| \mu(M)$ and

3) $det(I - M\Delta) \neq 0 \quad \forall\ \Delta \in \left\{ diag(\Delta_1, \Delta_2, \ldots, \Delta_n) \,\middle|\, \bar{\sigma}(\Delta_i) < 1 \right\}$

iff $\quad \mu(M) \leq 1$

Obviously, μ is a function of M which depends on the *structure* of $\left\{ \Delta \right\}$. To be precise, a multi-index could be constructed which would specify the structure of $\left\{ \Delta \right\}$ and μ would depend on that index. For this informal discussion, just keep in mind this fact and assume that a structure is specified. Clearly $\bar{\sigma}$ and ρ are special cases of μ for particular structures as indicated above. Furthermore, for any structure

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M). \tag{4}$$

Given these bounds, how important is μ? The answer can be clearly seen from the following examples:

Suppose $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ and consider

1) $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \rho(M)=0 \quad \bar{\sigma}(M)=1$

$det(I - M\Delta) = 1 \quad \text{so} \quad \mu(M) = 0$

2) $M = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \rho(M)=0 \quad \bar{\sigma}(M)=1$

$det(I - M\Delta) = 1 \div \dfrac{\delta_1 - \delta_2}{2} \quad \text{so} \quad \mu(M) = 1$

Thus neither ρ nor $\bar{\sigma}$ provide useful bounds even in simple cases. The only time they do provide reliable bounds is when $\rho \approx \bar{\sigma}$. Thus better bounds on μ are needed to pursue the problem in Case 3.

For the rest of the discussion fix a structure for the Δ's as

$$\underline{X} = \left\{ diag(\Delta_1, \Delta_2, \ldots, \Delta_n) \right\}. \tag{5}$$

Then

$$\mu(M) = \left\{ \min_{\Delta \in \underline{X}} \left[\bar{\sigma}(\Delta) \,\middle|\, det(I - M\Delta) = 0 \right] \right\}^{-1}. \tag{6}$$

This expression is little more than a definition of μ since the optimization problem implied by it is nonconvex, but it shows that μ exists as desired. To obtain useful properties of μ, some additional definitions are needed. Let

$$\underline{U} = \left\{ diag(U_1, U_2, \ldots, U_n) \,\middle|\, U_i^* U_i = I \right\} \tag{7}$$

$$\underline{D} = \left\{ diag(d_1 I, d_2 I, \ldots, d_n I) \,\middle|\, d_i \in \mathbb{R}+ \right\} \tag{8}$$

where the sets $\underline{U}$ and $\underline{D}$ match the structure of $\underline{X}$. Note that the $\underline{U}$ and $\underline{D}$ leave $\underline{X}$ invariant in the sense that

1) $\Delta \in \underline{X}$, $U \in \underline{U}$ imply $\bar{\sigma}(\Delta U) = \bar{\sigma}(U\Delta) = \bar{\sigma}(\Delta)$

2) $\Delta \in \underline{X}$, $D \in \underline{D}$ imply $D\Delta D^{-1} = \Delta$

From these two properties and the definition above expression for μ, one immediately obtains

$$\max_{U \in \underline{U}} \rho(MU) \le \mu(M) \le \inf_{D \in \underline{D}} \bar{\sigma}(DMD^{-1})$$

The first important theorem about μ is

Theorem 1 $\max_{U \in \underline{U}} \rho(MU) = \mu(M)$

This theorem expresses μ in terms of familiar linear algebraic objects. Unfortunately, the implied optimization problem is nonconvex so it does not immediately yield a computational approach. The second important theorem is

Theorem 2 If $n \le 3$ $\mu(M) = \inf_{D \in \underline{D}} \bar{\sigma}(DMD^{-1})$

This theorem states that if there are 3 or fewer blocks (no restriction on size), then $\mu(M)$ is just $\bar{\sigma}$ of a block diagonal similarity of M. Furthermore $\bar{\sigma}(DMD^{-1})$ is convex in D so that the infimum can be found by search over $n-1$ real parameters.

The theorem is not true for $n \ge 4$, but it is conjectured that $\inf_{D \in \underline{D}} \bar{\sigma}(DMD^{-1})$ still provides a reasonably tight bound for μ. Also, many problems of interest have 3 or fewer blocks so this provides a reasonable computational scheme.

Another important aspect of this theorem is that μ may be viewed as $\bar{\sigma}$ plus scaling. Thus the general synthesis methods recently developed to optimize the L_∞ norm (i.e. $\bar{\sigma}$) may be applied, via scalings, to optimize μ. This will be discussed more in the sybthesis section. Now back to Case 3.

1.3.3 SSV Analysis of Systems

Abuse notation and define

$$\left\|M\right\|_{\mu} = \sup_{\omega} \mu(M(j\omega)). \tag{1}$$

Although $\left\|\bullet\right\|_{\mu}$ is not a norm, this will be convenient. Recall that $\left\|M\right\|_{\mu}$ is a function of M which also depends on the assumed structure of the perturbations.

Case 3a involves stability in the presence of ***structured*** perturbations and the result is analogous with Case 2b. In fact, 3a reduces to 2b in the case that there is a single block in the perturbation. Suppose that $\Delta \in BRH_{\infty}$ and the Δ's have the structure $\Delta = diag(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n)$. The *GAT* for Case 3a is

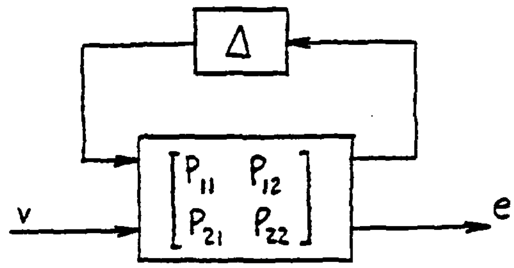
Theorem 1 $F_u(P,\Delta)$ is internally stable for all *structured* $\Delta \in BRH_{\infty}$ **iff** $\left\|P_{11}\right\|_{\mu} \le 1$

Case 3b puts everything together and is really the payoff for μ analysis. The problem is to check that $\left\|e\right\|_2 \le 1$ is satisfied for all $\left\|u\right\|_2 \le 1$ and all structured perturbations. Recall that from 2a and 2b that both stability with a single perturbation and performance with L_2 inputs involve the same test using $\left\|\bullet\right\|_{\infty}$ although on different parts of the system. This means that the system in Figure 1 has internal stability and $\left\|e\right\|_2 \le 1$ for all $\left\|u\right\|_2 \le 1$ and $\Delta = diag(\Delta_1, \Delta_2, \ldots, \Delta_n) \in BRH_{\infty}$ if and only if the system in Figure 2 has internal stability for all structured Δ and all $\Delta_{n+1} \in BRH_{\infty}$. This is exactly Case 3a with the structure $\tilde{\Delta} = diag(\Delta_1, \Delta_2, \ldots, \Delta_n, \Delta_{n+1})$. Using this structure for μ yields the following:

Theorem 2 $\mathbf{F}_u(P,\Delta)$ is internally stable and $\|e\|_2 \leq 1$ for all $\|u\|_2 \leq 1$ and $\Delta = diag(\Delta_1, \Delta_2, \ldots, \Delta_n) \in BRH_\infty$

iff $\|P\|_\mu \leq 1$

This is a remarkably useful theorem. It says that $\|P\|_\mu \leq 1$ implies not only stability for all structured perturbations but also that $\|e\|_2 \leq 1$ for all $\|u\|_2 \leq 1$ and all structured perturbations. Furthermore, $\|P\|_\mu > 1$ implies that there exists a u with $\|u\|_2 \leq 1$ and a structured Δ such that either $\|e\|_2 > 1$ or $\mathbf{F}_u(P,\Delta)$ is internally unstable. This is the first general result which guarantees performance for a whole *set* of plants and gives an exact (nonconservative) analysis test.



$$\Delta = \mathrm{diag}\,(\Delta_1, \ldots, \Delta_n)$$

Figure 1

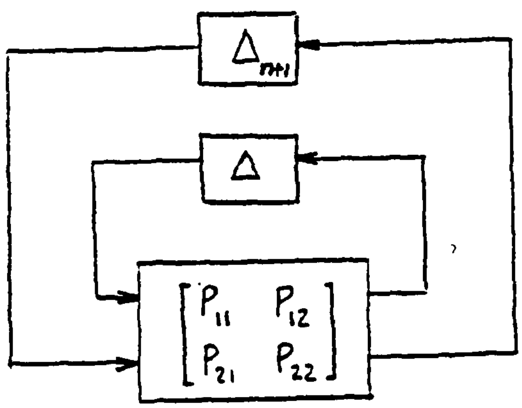


Figure 2

1.4 A Glimpse at Synthesis

This will be a sketchy outline of the new synthesis results. The details are somewhat complicated and are treated in Part 2 which is devoted to the synthesis theory. At this point, we simply want to point out how the analysis theory discussed in this part leads naturally to certain synthesis questions.

From the analysis results, we see that each case boils down to evaluating

$$\left\|P_{ij}\right\|_{\alpha} \qquad \alpha=2,\infty \text{ or } \mu \tag{1}$$

for some transfer function P_{ij}. Thus when the controller is put back into the problem, it involves just a simple linear fraction transformation as shown in the diagram in Figure 1. (Note: the P_{ij}'s here are not the same as the P_{ij}'s in the previous sections)

Each case then leads to the synthesis problem

$$\min_{K}\left\|\mathrm{F}_l(P,K)\right\|_{\alpha} \quad \text{for } \alpha=2,\infty, \text{ or } \mu \tag{2}$$

subject to internal stability of the nominal. Here

$$\mathrm{F}(P,K)=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}.$$

The solution of this problem for $\alpha=2$ and ∞ is the focus of Part 2 on Synthesis Theory. The solution presented there unifies the two approaches in a common synthesis framework. The $\alpha = 2$ case was already known and the results are simply a new interpretation. The $\alpha=\infty$ case had been solved only for special cases where P_{12} and P_{21} are square. Also, the existing solutions did not have computational schemes allowing their use on even moderately sized problems. These two limitations, especially the former, restricted the application of the pioneering H_∞ methods to fairly simple problems, such as sensitivity minimization. The new solution eliminates these two limitations.

Unfortunately, this new solution for the H_2 and H_∞ suffers from the same limitations imposed by restrictive assumptions about uncertainty as do the underlying analysis methods. While the SSV is a great improvement for analysis (Case 3), synthesis for the $\alpha=\mu$ case is not yet fully solved. Recalling that μ may be obtained by scaling and applying $\left\|\cdot\right\|_\infty$, a reasonable approach is to "solve"

$$\min_{K,D}\left\|DF(P,-K)D^{-1}\right\|_\infty \qquad (3)$$

by iteratively solving for K and D. With either K or D fixed, the global optimum in the other variable may be found using the μ and H_∞ solutions described previously. Example designs have been done and this scheme seems to work well, but global convergence is not guaranteed. In fact, a counterexample has been constructed where (3) reaches a local minimum which is not global.

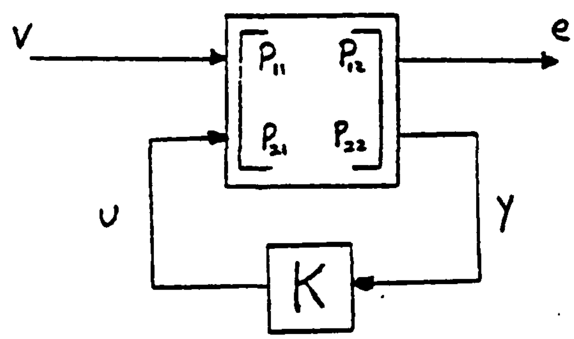


Figure 1

Part 2. Synthesis Theory

2.1 Introduction

2.1.1 Overview of Synthesis

From the previous part of these notes on analysis, we have seen that the synthesis problem in each case reduces to finding a controller K which achieves internal stability and solves

$$\min_{K \in R_p^{p_2 \times m_2}} \left\| F_l(P,K) \right\|_\alpha \qquad \alpha = 2, \infty, \text{ or } \mu \tag{1}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R_p^{(p_1+p_2)\times(m_1+m_2)} \quad , \quad P_{ij} \in R^{p_i \times m_j}$$

and

$$F_l(P,K) = P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}$$

We will restrict our attention for now to the $\alpha=2$ and ∞ cases of (1). Recall that the $\alpha=\mu$ case of (1) can be converted to the $\alpha=\infty$ case by scaling. The approach of these notes is to develop the $\alpha=2$ and ∞ cases in a parallel manner, emphasizing their common features.

We begin by considering the special case of (1) where all matrices are constants. This is an interesting problem in its own right and manages to capture the essential features of the general problem. While the $\alpha=2$ case is quite straightforward, the key step in the solution of (1) for $\alpha = \infty$ was first published in 1982 by Davis, Kahan, and Weinberger in their important paper on norm-preserving dilations [DKW]. The treatment of the constant case in this chapter is based on this paper.

The remainder of this part of the notes involves taking each step of the solution to (1) for the constant case and generalizing to the case of real-rational matrices. The difficulty arises from stability/causality considerations which are not present in the constant matrix case. If the requirement for internal stability were dropped from the optimization problem in (1) then it would reduce immediately to the constant case at each frequency. In general, however, any K obtained in this way would not be one such that $F_l(P,K)$ is stable.

The role of the stability requirement can be seen by considering a special case of (1) where $P_{22} = 0$ and both P_{12} and P_{21} are square and inner. With these assumptions, $F_l(P,K) = P_{11} + P_{12}KP_{21}$, so internal stability is equivalent to stability of K ($K \in RH_\infty$). Since both $\alpha = 2$ and ∞ norms are unitary (i.e. inner) invariant,

$$\left\|F_l(P,K)\right\|_\alpha = \left\|R + K\right\|_\alpha \text{ where } R = P_{12}^* P_{11} P_{21}^*. \tag{2}$$

Since the RH_∞ part of R in (2) may be absorbed into K, we see immediately that our simplified problem reduces to

$$\min_{K \in RH_\infty^{p_2 \times m_2}} \left\|R + K\right\|_\alpha \quad \alpha=2 \text{ or } \infty, \tag{3}$$

where $R \in RH_2^\perp$.

For the $\alpha=2$ case we have immediately that the optimal K in (3) is $K=0$, since L_2 is a Hilbert space and H_2 and $H_2^\perp$ are orthogonal subspaces. The $\alpha=\infty$ case is a version of the well-known matrix interpolation problem. It is the matrix generalization of the classical "best approximation" problem of finding the nearest H_∞ function to a given function in L_∞.

This best approximation problem has a rich history. The rational matrix version is as follows: given $R \in RH_2^\perp$ (i.e. strictly proper with all poles in the

open right half plane) find all $K \in RH_\infty$ (if any exist) such that

$$\|R - K\|_\infty \leq \gamma. \tag{4}$$

More general versions drop the rationality assumption. There is a vast literature on interpolation problems and the related question of best approximation. Some of the early references can be found in the seminal paper on "generalized interpolation" by Sarason [Sar]. Alternative treatments include those by Adamjan, Arov, and Krein ([AAK1]-[AAK3]) and Ball and Helton [BaH].

The most recent contribution to the rational matrix best approximation problem is by Glover, who gives a complete parametrization for all solutions to (4) in terms of a state-space *realization* for R [Glo]. Furthermore, a realization of the parametrization can be computed from the realization of R using standard real matrix operations. The main goal of this part on synthesis is to reduce the problem in (1) to that in (3). The $\alpha=2$ case is then trivial, and the $\alpha=\infty$ case can be solved using Glover's results.

An important feature of the results that follow that achieve the reduction of (1) to (3) is that each step can be computed using standard real matrix operations in terms of a realization of P. This allows the entire solution scheme to be implemented in software in a reasonably straightforward manner, which is important if the mathematical results of these notes are to be applied to engineering problems. The connection of the theory with computation is reinforced throughout by the approach of proving existence parts of theorems by giving what amounts to an algorithm for computing a particular solution. While this occasionally makes for rather tedious proofs, the fact that the theorem can be implemented almost directly makes this approach seem well worth it. It should be noted that each main theorem in this part

has been implemented and experimental control systems have been designed using this software. The results to date have been most encouraging.

To see the importance of the stability requirement and the resulting best approximation problem, note that relaxation of the requirement in (3) that $K \in RH_\infty$ to simply requiring that $K \in R_p$ means that the minimum of 0 in (3) is achieved by letting $K = -R$. This is essentially what holds for the constant case. Thus the stability requirement restricts the achievable optimal in (3). As we shall see, the stability requirement has a similar impact on the general problem in (1).

2.1.2 Constant Matrix Case

In this section, we will consider a special synthesis problem where all matrices are constants. The constant matrix case will allow us to study the synthesis problem in a simplified context, but one which parallels the rational case.

For constant matrices, the norms reduce to

$$\|P\|_\infty = \bar{\sigma}(P)$$

$$\|P\|_2 = \left[Tr(P^*P)\right]^{1/2}$$

Note that these definitions are not conventional, but they are convenient in allowing parallel development of the constant and rational cases.

Consider the constant matrix problem

$$\min_{K \in \mathbb{C}^{p_2 \times m_2}} \|\mathrm{F}_l(P,K)\|_\alpha \qquad \alpha = 2,\infty \tag{1}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(m_1+m_2)}, \quad P_{ij} \in \mathbb{C}^{p_i \times m_j},$$

and

$$\mathrm{F}_l(P,K) = P_{11} - P_{12}K(I-P_{22}K)^{-1}P_{21}.$$

Assume that $P_{12}^*P_{12} > 0$ and $P_{21}P_{21}^* > 0$.

The first step is to make the substitution of variables

$$K = \hat{Q}(I+P_{22}\hat{Q})^{-1}, \quad \hat{Q} = (P_{12}^*P_{12})^{-1/2}Q(P_{21}P_{21}^*)^{-1/2} \tag{2}$$

so

$$Q = (P_{12}^*P_{12})^{1/2}K(I-P_{22}K)^{-1}(P_{21}P_{21}^*)^{1/2} \tag{3}$$

Using the linear fractional representation notation,

2.1.5 Rational Matrix Generalization

The steps in the rational case closely parallel the constant case, as shown in Figure 1. Most of the work in the remaining chapters is devoted to generalizing these steps from constants to rationals. The source of all the difficulty in the rational case comes from the requurement for internal stability, or equivalently, causality. Without this the rational case would reduce to the constant case at each frequency, and could be solved using the results of the previous two sections.

We will now outline the steps required to solve the rational case and preview the upcoming chapters which complete the details.

1) **Parametrization:** Find J so that the substitution $K=\mathbf{F}_l(J,Q)$ yields

$$\begin{aligned} \mathbf{F}_l(P,K) &= \mathbf{F}_l(P,\mathbf{F}_l(J,Q)) \\ &= \mathbf{F}_l(T,Q) \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \tag{10}$$

with the additional requirement that $T \in H_\infty$ and

$$\begin{aligned} &\mathbf{F}_l(P,K) \quad \text{internally stable} \\ &\qquad \text{iff} \quad Q \in H_\infty. \end{aligned} \tag{11}$$

This parametrizes all stabilizing K's in terms of a stable $Q \in H_\infty$ in addition to providing an affine parametrization of all stable $\mathbf{F}_l(P,K)$. This so-called "Youla parametrization" [You2] is developed in Chapter 2 on Stabilization.

A further requirement is that T_{12} and T_{21} be inner, that is $T_{12}^* T_{12}=I$ and $T_{21}T_{21}^*=I$. Methods for obtaining the particular parametrizations which

2.1.4 **Figure 1**

$$\min_K \left\| F_l(P,K) \right\|_\alpha \quad \text{where} \quad F_l(P,K) = P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}$$

$$\Big\downarrow \quad \text{parametrization} \quad K = F_l(J,Q)$$

$$\min_Q \left\| T_{11} - T_{12}QT_{21} \right\|_\alpha \quad \text{where} \quad T_{12}^*T_{12} = I \qquad T_{21}T_{21}^* = I$$

$$\Big\downarrow \quad \text{unitary invariance}$$

$$\min_Q \left\| \begin{matrix} R_{11}-Q & R_{21} \\ R_{12} & R_{22} \end{matrix} \right\|_\alpha$$

α=2 projection

α=∞ dilation

$$\min_{\hat{Q}} \left\| G-\hat{Q} \right\|_\alpha$$

$$\Big\downarrow$$

$$Q_{opt}$$

$$\Big\downarrow \quad K_{opt} = F_l(J, Q_{opt})$$

$$K_{opt}$$

Note that $Q=-R_{11}$ is one solution to (6) and (8).

4) **Recovery of the optimal** K: This is obtained by simply computing K from the formula $K=F_l(J,Q)$ used in step 1) to parametrize the problem.

$$= \begin{bmatrix} T_{12}^* \\ T_{\perp}^* \end{bmatrix} T_{11} \begin{bmatrix} T_{21}^* & \tilde{T}_{\perp}^* \end{bmatrix} \tag{3}$$

Recall that without loss of generality, we may assume the $T_{11}\tilde{T}_{\perp}^*=0$ so that (2) becomes

$$\begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix}. \tag{4}$$

3) **Projection / Dilation:** At this point the $\alpha=2$ and $\alpha=\infty$ cases differ. For $\alpha=2$, the problem reduces, by projection, to

$$\min_{Q} \left\| R_{11} + Q \right\|_2 \tag{5}$$

which has the unique solution $Q=-R_{11}$.

The $\alpha=\infty$ case must be treated using the matrix dilation theory of the previous section. Recall that, in general, the solution is not unique. From Theorem 3.1, all solutions to

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_{\infty} \le \gamma \quad \text{for} \quad \gamma \ge \left\| R_{21} \right\|_{\infty} \tag{6}$$

are of the form

$$Q = -R_{11} + Y(\gamma^2 I - R_{21}^* R_{21})^{1/2} \tag{7}$$

for some $\left\|Y\right\|_{\infty} \le 1$. Corollary 3.1 gave the alternative characterization that

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_{\infty} \le \gamma \quad \text{for} \quad \gamma > \left\| R_{21} \right\|_{\infty} \tag{8}$$

if and only if

$$\left\| (R_{11}+Q)(\gamma^2 I - R_{21}^* R_{21})^{-1/2} \right\|_{\infty} \le 1. \tag{9}$$

It is this latter characterization which will be used in the rational case.

2.1.4 Summary of Constant Problem

The rational matrix problem in equation (1.1) can be solved in a manner which parallels the treatment of the constant case in the last two sections. This generalization is the focus of the next three chapters on synthesis. To reinforce the similarity between the constant and rational case, we will now review the key steps from the previous two sections and preview their generalizations to the rational case.

Consider the diagram in Figure 1. This summarizes the steps in the constant matrix problem (2.1). The main steps are as follows:

1) **Parametrization:** Make the substitution $K=F_l(J,Q)$ so that

$$\begin{aligned} F_l(P,K) &= F_l(P,F_l(J,Q)) \\ &= F_l(T,Q) \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \qquad (1)$$

is affine. Additionally, we want $T_{12}^*T_{12}=I$ and $T_{21}T_{21}^*=I$.

2) **Unitary Invariance:** Find $T_\perp$ and $\tilde{T}_\perp$ so that $\begin{bmatrix} T_{12} & T_\perp \end{bmatrix}$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}$ are square and unitary.

Pre- and post-multiply by $\begin{bmatrix} T_{12} & T_\perp \end{bmatrix}^*$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}^*$ to yield

$$\begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \qquad (2)$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

$$\left|\left|\left[R_2 + U_2Q\right]\right|\right|_\infty \le 1 \tag{26}$$

where $R_2 = (\gamma^2 I - AA^*)^{-1/2} R_1$

$U_2 = (\gamma^2 I - AA^*)^{-1/2} U.$

To complete this, simply factor U_2 to extract a unitary factor, and apply the dual of (22)-(26) to (26). Although the formulas get messy, (22) can be solved in this manner.

In each of these cases, Theorem 2, Theorem 3, their corollaries, and the solution described above, the general case reduces almost immediately to application of Theorem 1 or Corollary 1. Thus, when it is convenient, we will consider (4) rather than (16) and (26) rather than (22). This will simplify the discussion of the rational case without introducing any loss of generality.

The restriction that $\gamma > \gamma_o$ in Corollary 3 and (22)-(26) does introduce some loss of generality. If these alternatives to Theorem 3 are used, it is not possible to get all solutions for $\gamma = \gamma_o$. All solutions arbitrarily close to the optimal γ_o is the best that can be done. The reason for considering this special case is that (20)-(21) and (22)-(26) have reasonably straighforward generalizations to the rational matrix case, whereas (17)-(18) do not. A further difficulty with the rational case is that, unlike the constant case, it appears that it is not possible to actually compute γ_o exactly. This makes our inability to obtain a direct generalization of Theorem 3 seem less critical, at least with respect to application of this theory. More will be said about this in Section 5, but the rational matrix generalization will focus on (22)-(26).

The following corollary gives an alternative version of Theorem 3.

Corollary 3 For $\gamma > \gamma_o$,

$$\left\| \begin{bmatrix} X & B \\ C & A \end{bmatrix} \right\|_\infty \le \gamma \tag{19}$$

iff

$$\left\| (I - YY^*)^{-1/2}(X + Y^*AZ)(I - Z^*Z)^{-1/2} \right\|_\infty \le \gamma \tag{20}$$

where

$$\begin{aligned} Y &= B(\gamma^2 I - A^*A)^{-1/2} \\ Z &= (\gamma^2 I - AA^*)^{-1/2} C \end{aligned} \tag{21}$$

There are many alternative characterizations of solutions to (19), although the formulas in (20) and (21) seem to be the simplest.

For the problem in (14), the following equivalences apply for all $\gamma > \gamma_o$:

$$\left\| R + UQV \right\|_\infty \le \gamma \tag{22}$$

iff

$$\left\| \begin{bmatrix} RV^* + UQ & RV_\perp^* \end{bmatrix} \right\|_\infty \le \gamma \tag{23}$$

iff

$$\left\| \begin{bmatrix} R_1 - UQ & A \end{bmatrix} \right\|_\infty \le \gamma \tag{24}$$

where $R_1 = RV^*$ and $A = RV_\perp^*$

iff

$$\left\| (\gamma^2 I - AA^*)^{-1/2} \begin{bmatrix} R_1 + UQ \end{bmatrix} \right\|_\infty \le 1 \tag{25}$$

(by Corollary 1′)

iff

form of (2.4) for the problem

$$\gamma_o = \min_Q \left\| R + UQV \right\|_\infty \tag{14}$$

where $U^*U = I$ and $VV^* = I$

Corollary 2

$$\gamma_o = \max \left\{ \left\| U_\perp^* R \right\|_\infty , \left\| R V_\perp^* \right\|_\infty \right\} \tag{15}$$

The following theorem parametrizes all solutions to (1). The proof is omitted (it can be found in [DHK]), but is similar to Theorem 2 and involves application of Theorem 1 and 1'.

Theorem 3 Suppose $\gamma \geq \gamma_o$. The solutions X such that

$$\left\| \begin{bmatrix} X & B \\ C & A \end{bmatrix} \right\|_\infty \leq \gamma \tag{16}$$

are exactly those of the form

$$X = -YA^*Z + \gamma(I-YY^*)^{1/2} W (I-Z^*Z)^{1/2} \tag{17}$$

where W is an arbitrary contraction ($\left\| W \right\|_\infty \leq 1$) and Y and Z solve the linear equations

$$\begin{aligned} B &= Y(\gamma^2 I - A^*A)^{1/2} \\ C &= (\gamma^2 I - AA^*)^{1/2} Z. \end{aligned} \tag{18}$$

Theorem 2

$$\gamma_o = \max \left\{ \left\| \begin{bmatrix} C & A \end{bmatrix} \right\|_\infty, \left\| \begin{bmatrix} B \\ A \end{bmatrix} \right\|_\infty \right\} \tag{13}$$

Proof: Denote by $\hat{\gamma}$ the right hand side of the equation (13). Clearly, $\gamma_o \geq \hat{\gamma}$ since compressions are norm decreasing. That $\gamma_o \leq \hat{\gamma}$ will be shown by using Theorem 1 and 1'.

From Theorem 1 we have that $B = Y(\hat{\gamma}^2 I - A^*A)^{1/2}$ for some Y such that $\|Y\|_\infty \leq 1$. Similarly, Theorem 1' yields $C = (\hat{\gamma}^2 I - AA^*)^{1/2} Z$ for some Z with $\|Z\|_\infty \leq 1$.

Let $\hat{X} = -YA^*Z$. Then

$$\begin{aligned} \left\| \begin{bmatrix} \hat{X} & B \\ C & A \end{bmatrix} \right\|_\infty &= \left\| \begin{bmatrix} -YA^*Z & Y(\hat{\gamma}^2 I - A^*A)^{1/2} \\ (\hat{\gamma}^2 I - AA^*)^{1/2} Z & A \end{bmatrix} \right\|_\infty \\ &\leq \left\| \begin{bmatrix} -A^* & (\hat{\gamma}^2 I - A^*A)^{1/2} \\ (\hat{\gamma}^2 I - AA^*)^{1/2} & A \end{bmatrix} \right\|_\infty \\ &= \hat{\gamma} \end{aligned}$$

Since

$$\begin{bmatrix} -A^* & (\hat{\gamma}^2 I - A^*A)^{1/2} \\ (\hat{\gamma}^2 I - AA^*)^{1/2} & A \end{bmatrix} \begin{bmatrix} -A & (\hat{\gamma}^2 I - AA^*)^{1/2} \\ (\hat{\gamma}^2 I - A^*A)^{1/2} & A^* \end{bmatrix} = \begin{bmatrix} \hat{\gamma}^2 I & 0 \\ 0 & \hat{\gamma}^2 I \end{bmatrix}$$

Thus $\hat{\gamma} \geq \gamma_o$, so $\hat{\gamma} = \gamma_o$.

This theorem gives one solution to (12) and an expression for γ_o. As in (3), thei may be more than one solution to (12), although Theorem 2 only exhibits one. Theorem 3 considers the problem of parametrizing all solutions. The solution $\hat{X} = -YA^*Z$ is the "central" solution analogous to $X = 0$ in (3). The next corollary is an alternative statement of Theorem 2 using the

Corollary 1: For $\gamma > \gamma_o$,

$$\left\| \begin{bmatrix} X \\ A \end{bmatrix} \right\|_\infty \le \gamma \tag{6}$$

iff

$$\left\| X(\gamma^2 I - A^* A)^{-1/2} \right\|_\infty \le 1. \tag{7}$$

The corresponding dual results are

Theorem 1' For $\forall \gamma \ge \gamma_o$

$$\left\| \begin{bmatrix} X & A \end{bmatrix} \right\|_\infty \le \gamma \;\text{ iff }\; \exists Y,\ \left\| Y \right\|_\infty \le 1 \tag{8}$$

such that

$$X = (\gamma^2 I - AA^*)^{1/2} Y \tag{9}$$

Corollary 1' For $\gamma > \gamma_o$

$$\left\| \begin{bmatrix} X & A \end{bmatrix} \right\|_\infty \le \gamma \tag{10}$$

iff

$$\left\| (\gamma^2 I - AA^*)^{-1/2} X \right\|_\infty \le 1 \tag{11}$$

Now, returning to the problem in (1), let

$$\gamma_o = \min_X \left\| \begin{bmatrix} X & B \\ C & A \end{bmatrix} \right\|_\infty \tag{12}$$

The following theorem, usually attributed to Parrott [Par], will play a central role in the synthesis theory. The proof is a straightforward application of Theorem 1 and 1'.

in (3), it is immediate that $\gamma_o = \|A\|_\infty$. The following theorem characterizes all solutions to (3).

Theorem 1: For $\forall \gamma \geq \gamma_o$,

$$\left\| \begin{bmatrix} X \\ A \end{bmatrix} \right\|_\infty \leq \gamma \tag{4}$$

iff $\exists\, Y$ with $\|Y\|_\infty \leq 1$ such that

$$X = Y(\gamma^2 I - A^* A)^{1/2} \tag{5}$$

Proof:

$$\left\| \begin{bmatrix} X \\ A \end{bmatrix} \right\|_\infty \leq \gamma$$

iff

$$X^* X - A^* A \leq \gamma^2 I$$

iff

$$X^* X \leq (\gamma^2 I - A^* A)$$

iff

$$\|Xu\| \leq \|(\gamma^2 I - A^* A)^{1/2} u\| \qquad \forall u$$

iff

$$X = Y(\gamma^2 I - A^* A)^{1/2} \quad \text{for some} \quad \|Y\|_\infty \leq 1$$

This theorem implies that, in general, (3) has more than one solution. This is in contrast to the $\alpha = 2$ case. The solution $X = 0$ is the central solution but others are possible unless $A^* A = \gamma^2 I$. A more restricted version of the theorem is

2.1.3 Matrix Dilation Problems

Consider the optimization problem

$$\min_X \left\| \begin{bmatrix} X & B \\ C & A \end{bmatrix} \right\|_\infty \tag{1}$$

where X, B, C, A are constant matrices of compatible dimensions. This is a restatement of (2.7) for the $\alpha=\infty$ case. The matrix $\begin{bmatrix} X & B \\ C & A \end{bmatrix}$ is a *dilation* of its submatrices as indicated in the following diagram:

$$\begin{tikzcd}[ampersand replacement=\&, row sep=huge, column sep=huge]
\begin{bmatrix} X & B \\ C & A \end{bmatrix} \arrow[r, shift left, "c"] \arrow[d, shift left, "c"] \& \begin{bmatrix} B \\ A \end{bmatrix} \arrow[l, shift left, "d"] \arrow[d, shift left, "c"] \\
\begin{bmatrix} C & A \end{bmatrix} \arrow[u, shift left, "d"] \arrow[r, shift left, "c"] \& \begin{bmatrix} A \end{bmatrix} \arrow[l, shift left, "d"] \arrow[u, shift left, "d"]
\end{tikzcd} \tag{2}$$

In this diagram, c stands for the operation of *compression* and d stands for *dilation*. Compression is always norm decreasing; sometimes dilation can be made to be norm preserving. Norm preserving dilations are the focus of this section, which basically follows the development in [DKW] and [Pow]. See [DKW] for additional references on dilation problems.

The simplest matrix dilation problem occurs when solving

$$\min_X \left\| \begin{bmatrix} X \\ A \end{bmatrix} \right\|_\infty \tag{3}$$

Although (3) is a much simplified version of (1), we will see that it contains all the essential features of the problem. Letting γ_o denote the minimum norm

$$\min_{Q \in \mathfrak{C}} \left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\alpha} \tag{8}$$

and solution of (8) yields a solution of (1) by solving (2) for K.

The $\alpha = 2$ case can be solved immediately from (8) since

$$\left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2^2 = \left\| R_{11}+Q \right\|_2^2 + \left\| \begin{bmatrix} 0 & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2^2. \tag{9}$$

Thus

$$Q_{opt} = -R_{11} = -T_{12}^{*} T_{11} T_{21}^{*}$$

and

$$\min_{Q \in \mathfrak{C}} \left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0 & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2 \tag{10}$$

The simplicity of the $\alpha=2$ case is responsible for much of its appeal. Optimization in this norm reduces to projection since L_2 is a Hilbert space. This holds as well for the rational matrix problem.

The $\alpha=\infty$ case is somewhat more complicated since L_∞ is not a Hilbert space and the minimization in (8) cannot be solved by projection. Fortunately, L_∞ arises as the space of linear operators on the Hilbert space L_2, and (8) can be treated as a dilation problem. The next section focuses on matrix dilation problems.

$$K = \mathbf{F}_l(J, Q) \tag{4}$$

where

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 0 & (P_{12}^* P_{12})^{-1/2} \\ (P_{21} P_{21}^*)^{-1/2} & -(P_{21} P_{21}^*)^{-1/2} P_{22} (P_{12}^* P_{12})^{-1/2} \end{bmatrix}$$

With this substitution, we have

$$\begin{aligned} \mathbf{F}_l(P, K) &= \mathbf{F}_l(P, \mathbf{F}_l(J, Q)) \\ &= P_{11} + \left[P_{12}(P_{12}^* P_{12})^{-1/2}\right] Q \left[(P_{21} P_{21}^*)^{-1/2} P_{21}\right] \\ &= T_{11} + T_{12} Q T_{21} \end{aligned} \tag{5}$$

where the T_{ij} are defined in the obvious way. This parametrization has converted the nonlinear problem in (1) to one affine in the parameter Q. Note that $T_{12}^* T_{12} = I$ and $T_{21} T_{21}^* = I$. Thus we can find $T_\perp$ and $\tilde{T}_\perp$ such that both $\begin{bmatrix} T_{12} & T_\perp \end{bmatrix}$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}$ are square and unitary.

Since both $\alpha = 2$ and ∞ norms are unitary invariant

$$\begin{aligned} \left\| T_{11} + T_{12} Q T_{21} \right\|_\alpha &= \left\| T_{11} + \begin{bmatrix} T_{12} & T_\perp \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix} \right\|_\alpha \\ &= \left\| \begin{bmatrix} T_{12} & T_\perp \end{bmatrix}^* T_{11} \begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}^* + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_\alpha \\ &= \left\| \begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\alpha \end{aligned} \tag{6}$$

where

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} T_{12}^* T_{11} T_{21}^* & T_{12}^* T_{11} \tilde{T}_\perp^* \\ T_\perp^* T_{11} T_{21}^* & T_\perp^* T_{11} \tilde{T}_\perp^* \end{bmatrix} \tag{7}$$

Thus the problem in (1) reduces to

achieve this are developed in Chapter 3 on Factorization.

2) **Unitary Invariance:** Find $T_\perp$ and $\tilde{T}_\perp$ so that $\begin{bmatrix} T_{12} & T_\perp \end{bmatrix}$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}$ are *square* and inner (also Chapter 3). Then pre- and post-multiply by $\begin{bmatrix} T_{12} & T_\perp \end{bmatrix}^*$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}^*$ to yield

$$\begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{12}$$

where

$$R = \begin{bmatrix} T_{12}^* \\ T_\perp^* \end{bmatrix} T_{11} \begin{bmatrix} T_{21}^* & \tilde{T}_\perp^* \end{bmatrix}$$

Again, to simplify the presentation suppose that $T_{11}\tilde{T}_\perp^*=0$ so that (12) becomes

$$\begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix}. \tag{13}$$

3) **Projection / Dilation:** At this point the $\alpha=2$ and $\alpha=\infty$ cases again differ. For $\alpha=2$, the problem reduces, by projection, to

$$\min_{Q \in H_\infty} \left\| R_{11} + Q \right\|_2 \tag{14}$$

But since $R_{11} \in L_\infty$, $Q=-R_{11}$ would not correspond to a stable solution. The unique solution is yet another projection

$$Q_{opt} = P_{H_2}(R_{11}) \tag{15}$$

where P_{H_2} denotes projection onto H_2. When viewed appropriately, these two projections can be seen as a single projection onto a subspace of $L_2(j\mathbb{R}, \mathbb{C}^{p_1 \times m_1})$.

The $\alpha=\infty$ case is again treated as a dilation problem. While it is not true

in general that

$$\min_{Q \in H_\infty} \left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty > \hat{\gamma} \triangleq \left\| R_{21} \right\|_\infty \tag{16}$$

it is convenient to use the characterization in Corollary 3.1 and (4.8)-(4.9). Recall that

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty \leq \gamma \quad \text{for} \quad \gamma > \hat{\gamma} \tag{17}$$

iff

$$\left\| (R_{11}+Q)(\gamma^2 I - R_{21}^* R_{21})^{-1/2} \right\|_\infty \leq 1 \tag{18}$$

The key to proceeding in the rational case is to find $M \in RH_\infty$ such that $M^{-1} \in RH_\infty$ and $M^*M = (\gamma^2 I - R_{21}^* R_{21})$. If we use the symbol $(\gamma^2 I - R_{21}^* R_{21})^{1/2}$ to denote this M, then (18) makes sense in the rational case. Finding M involves spectral factorization and is treated in Chapter 3.

Given $M \in RH_\infty$ with the desired properties, (18) reduces to

$$\left\| G + \hat{Q} \right\|_\infty \leq 1 \tag{19}$$

where $G = R_{11} M^{-1} \in RH_\infty$ and $\hat{Q} = QM^{-1}$. Solving (19) for $\hat{Q} \in RH_\infty$ solves (18) for $Q \in RH_\infty$. Note that $Q = \hat{Q}M$ is in RH_∞ if $\hat{Q}$ is, since $M \in RH_\infty$ by construction.

The final step in the rational case then involves solving (19) for $\hat{Q} \in RH_\infty$. This is the standard mathematical problem of approximating an L_∞ matrix by an H_∞ matrix. The approach which fits most naturally with the methods of these notes is due to Glover [Glo].

Glover gives an explicit parametrization of all solutions to (19) in terms of a realization of G. When combined with the steps developed in these

notes which reduce

$$\left\|\mathbf{F}_l(P,K)\right\|_\infty \le \gamma \tag{20}$$

to (17) and then (19), this yields a parametrization of all K's which achieve internal stability and satisfy (20). The remainder of these notes is concerned with developing the mathematics to carry out the steps outlined above.

4) **Recovery of the optimal K:** Just as in the constant case $K_{opt} = \mathbf{F}_l(J, Q_{opt})$. This K_{opt} will stabilize $\mathbf{F}_l(P, K_{opt})$ since the parametrization in Step 1) insured that Q stable lead to internal stability of $\mathbf{F}_l(P,K) = \mathbf{F}_l(P, \mathbf{F}_l(J,Q))$.

Note that although restricting to the special case of (13) from (12) introduces no loss of generality, it may be possible that (16) does not hold. Let

$$\gamma_o \triangleq \min_{Q \in H_\infty} \left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty \tag{21}$$

$$\hat{\gamma} \triangleq \left\| R_{21} \right\|_\infty . \tag{22}$$

Then, it is possible that $\gamma_o = \hat{\gamma}$. In that case, the algorithm described above cannot be used to obtain all solutions such that

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty \le \gamma_o \tag{23}$$

since (17) is not equivalent to (18). It is possible to find all solutions for any γ arbitrarily close to γ_o.

A more fundamental difficulty with the algorithm outlined above is that it gives no simple way of determining γ_o, so that the algorithm must be iterated on with a search on γ. It seems that the best that can be expected is to get arbitrarily close to γ_o. In view of this, the special case when $\gamma_o = \hat{\gamma}$

would seem not to be of particular importance.

The question naturally arises whether the inability to compute γ_o is intrinsic or simply an artifice of the chosen methodology. This cannot be fully resolved in these notes, but there is some evidence to suggest that computation of γ_o is a fundamentally difficult problem. In particular, it can be shown that γ_o is equal to the norm of the operator

$$\begin{bmatrix} P_{H_2^\perp} R_{11} \\ R_{21} \end{bmatrix} P_{H_2} : L_2 \rightarrow L_2 \oplus L_2. \tag{24}$$

This implies that γ_o can be found as the solution of an eigenvalue problem, but of a matrix (or operator) which is infinite rank. Thus again, from this point of view, γ_o can be computed only to within an arbitrarily small accuracy, but not exactly. It is clear that this issue needs more research.

The remainder of these notes completes the details of the solution outlined in steps 1)-4) above. This provides a general solution to the H_∞ optimal control problem modulo the above remarks.

2.1.5 Figure 1

$$\min_{K \in P_p} \left\{ \left\| F_l(P,K) \right\|_\alpha \;\middle|\; \text{internal stability} \right\}$$

$$\Big\downarrow \;\; \begin{array}{l} \text{parametrization} \\ K = F_l(J,Q) \end{array}$$

$$\min_{Q \in H_\infty} \left\| T_{11} + T_{12} Q T_{21} \right\|_\alpha \quad \text{where} \quad T_{12}^* T_{12} = I \qquad T_{21} T_{21}^* = I$$

$$\Big\downarrow \;\; \text{unitary invariance}$$

$$\min_{Q \in H_\infty} \left\| \begin{matrix} R_{11} - Q & R_{21} \\ R_{12} & R_{22} \end{matrix} \right\|_\alpha$$

$$\begin{array}{cc} \alpha = 2 & \alpha = \infty \\ \text{projection} \searrow & \swarrow \text{dilation} \end{array}$$

$$\min_{Q \in H_\infty} \left\| G - \hat{Q} \right\|_\infty , \quad G \in L_\infty$$

$$\begin{array}{cc} \alpha = 2 & \alpha = \infty \\ \text{projection} \searrow & \swarrow \text{dilation} \end{array}$$

$$Q_{opt}$$

$$\Big\downarrow \;\; K_{opt} = F_l(J, Q_{opt})$$

$$K_{opt}$$

2.2 Stabilization

2.2.1 Introduction

This chapter considers the problem of finding a parametrization of all controllers $K \in R_p^{p_2 \times m_2}$ which achieve internal stability of

$$\mathbf{F}_l(P,K) = P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21} \tag{1}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R_p^{(p_1+p_2)\times(m_1+m_2)} \quad , \; P_{ij} \in R^{p_i \times m_j} \tag{2}$$

The notion of internal stability for (1) is considered in Section 2, where the appropriate definitions and properties are developed. The treatment here is in the spirit of [DeC] and [Per].

The approach taken in the rest of this chapter is to find J so that the substitution $K=\mathbf{F}_l(J,Q)$ yields

$$\begin{aligned} \mathbf{F}_l(P,K) &= \mathbf{F}_l(P,\mathbf{F}_l(J,Q)) \\ &= \mathbf{F}_l(T,Q) \\ &= T_{11} + T_{12}QT_{21} \end{aligned} \tag{3}$$

with the additional property that $T \in H_\infty$ and

$$\begin{aligned} &\mathbf{F}_l(P,K) \quad \text{internally stable} \\ &\quad \text{iff} \;\; Q \in H_\infty. \end{aligned} \tag{4}$$

This approach parametrizes all stabilizing K's in terms of a stable $Q \in H_\infty$ in addition to providing an affine parametrization of all stable $\mathbf{F}_l(P,K)$. This so-called "Youla parametrization" [You2] is developed in Section 3 on Parametrization of All Stabilizing Controllers. Section 3 puts the algebraic methods of stabilization as expounded by Desoer and co-authors [DML] in the context of the linear fractional transformations used in these

notes.

There are two main approaches to constructing stabilizing controllers of linear systems, the Youla parametrization and state-space methods using observers and state feedback([KBF],[Lue]). Each is well-known within the control theory community and each has its advantages. As indicated above, the Youla parametrization yields *all* stabilizing controllers as well as a convenient affine parametrization of the closed-loop system. Unfortunately, the standard algebraic treatment of this subject gives no reliable scheme to compute the coefficients of the parametrization. Observer-based stabilizing controllers, on the other hand, are easily constructed in terms of a realization of P using a variety of state-space computation schemes.

Section 4 shows that these two methods of stabilization are actually equivalent in a very direct way. This allows for the Youla parametrization to be constructed using the standard state-space computations of observer-based stabilization methods, providing explicit realizations of the desired J in (3) in terms of a realization of P. Section 5 puts all of the results of the preceding sections together to construct the desired affine parametrization of the closed loop system.

It should be noted that many of the results on the connections between the algebraic and observer-based stabilization methods were discovered independently by Nett and coauthors [NeJ]. Also, many of these results were known within the "systems over rings" community [Kha]. What is clearly original to these notes is the complete equivalence of the two stabilization methods (Section 4, Theorem 2) and the parametrization in terms of the general framework using linear fractional transformations (Section 5). Nevertheless, this material is not the main focus of these notes but is

presented in detail so that it may be used in developing the factorization methods of the next chapter.

2.2.2 Internal Stability

In this section P and K are fixed proper transfer matrices. The block diagram of Figure 1 represents the two equations

$$\begin{bmatrix} e \\ y \end{bmatrix} = P \begin{bmatrix} v \\ u \end{bmatrix}, \quad u = Ky. \tag{1}$$

Partition P accordingly:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{2}$$

It is convenient to introduce two fictitious external signals, w_1 and w_2, as in Figure 1a.

Suppose the signals v, w_1, and w_2 are specified and that u in Figure 1a is well-defined. Then so are e and y. Thus it makes sense to define the system in Figure 1a to be *well-posed* provided the transfer matrix from $\begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}$ to u exists and is proper.

Lemma 1. The system is well-posed if and only if

$$I - K(\infty)P_{22}(\infty) \quad \text{is invertible.} \tag{3}$$

Proof. Figure 1a implies the equations

$$u = w_1 + KY + Kw_2$$
$$y = P_{21}v + P_{22}u$$

and these in turn imply that

$$(I-KP)u = w_1 + KP_{21}v + Kw_2.$$

Thus well-posedness is equivalent to the condition that $(I-KP)^{-1}$ exists and is proper.

QED

It is straightforward to show that (3) is equivalent to either of the following two conditions:

$$\begin{bmatrix} I & -K(\infty) \\ -P_{22}(\infty) & I \end{bmatrix} \quad \text{is invertible ;} \tag{4}$$

$$I - P_{22}(\infty)K(\infty) \quad \text{is invertible .} \tag{5}$$

The well-posedness condition is simple to state in terms of state-space realizations. Introduce minimal realizations of P and K:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{6}$$

$$K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]. \tag{7}$$

Note that the partition in (6) corresponds to that in (2), i.e.,

$$P_{ij} = \left[\begin{array}{c|c} A & B_j \\ \hline C_i & D_{ij} \end{array}\right]. \tag{8}$$

Then $P_{22}(\infty)=D_{22}$ and $K(\infty)=\hat{D}$, so for example, from (4) well-posedness is equivalent to the condition that

$$\begin{bmatrix} I & -\hat{D} \\ -D_{22} & I \end{bmatrix} \quad \text{is invertible.} \tag{9}$$

Well-posedness will be assumed for the rest of this section. Let x and $\hat{x}$ denote the state vectors for P and K respectively, and write the system equations in Figure 1 with v set to zero and e ignored:

$$\dot{x} = Ax + B_2u \tag{10a}$$

$$y = C_2x + D_{22}u \tag{10b}$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y \tag{10c}$$

$$u = \hat{C}\hat{x} + \hat{D}y. \tag{10d}$$

The system of Figure 1 is *internally stable* provided the origin $(x,\hat{x}) = (0,0)$ is asymptotically stable. To get a concrete characterization of internal stability, solve equations (10b) and (10d) for u and y:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & -\hat{D} \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

(Note that the inverse exists from (9)). Now substitute this into (10a) and (10c) to get

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \tilde{A}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C_2 & 0 \end{bmatrix}.$$

Thus internal stability is equivalent to the condition that $\tilde{A}$ has all its eigenvalues in the open left half-plane.

It is routine to verify that the above definition of internal stability depends only on P and K, not specific realizations of them. The following result is standard.

Lemma 2. Consider a minimal realization of P as in (6). There exists a proper K achieving internal stability iff (A,B_2) is stabilizable and (C_2,A) is detectable.

The latter stabilizability and detectability conditions are *ssumed* throughout the remainder of this chapter. Since

$$P_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]. \tag{11}$$

equations (10) constitute a state-space representation of the system in Figure 2. Although the realization in (11) is not necessarily minimal, it is stabilizable and detectable, and these are enough to yield the following result.

Lemma 3. The system in Figure 1 is internally stable iff the one in Figure 2 is.

The next section contains a parametrization of all K's which achieve internal stability for the system in Figure 2. To simplify notation, define

$$G := P_{22}, \quad B := B_2, \quad C := C_2, \quad D := D_{22}.$$

Then (A,B) is stabilizable, (C,A) is detectable, and the system under study is that in Figure 3.

The above notion of internal stability is defined in terms of state-space realizations of G and K. It is important and useful to characterize internal stability from an input/output point of view. For this, consider the feedback system in Figure 4. This system is described by:

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{10}$$

Now it is intuitively clear that if the system in Figure 4 is internally stable then for all bounded inputs (v_1, v_2), the outputs (e_1, e_2) are also bounded. The following lemma shows that this idea lends to an input/output characterization of internal stability.

Lemma 4. The system in Figure 4 is internally stable if and only if $(I-GK)$ is invertible and the transfer matrix

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I+K(I-GK)^{-1}G & K(I-GK)^{-1} \\ (I-GK)^{-1}G & (I-GK)^{-1} \end{bmatrix} \tag{11}$$

between (v_1,v_2) and (e_1,e_2) belongs to RH_∞.

Proof. As above let (A,B,C,D) and $(\hat{A},\hat{B},\hat{C},\hat{D})$ be stabilizable and detectable realizations of G and K respectively. Then the state-space equations for the system in Figure 4 are:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$e_1 = v_1 + y_2, \quad e_2 = v_2 + y_1.$$

The last two equations can be rewritten as

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Now suppose that this system is internally stable. Then (7) implies that $(I-D\hat{D}) = (I-GK)(\infty)$ is invertible. Hence $(I-GK)$ is invertible. Further, since the eigenvalues of

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}$$

are in the open left half plane, it follows that the transfer matrix in (10) from (v_1,v_2) to (e_1,e_2) is in RH_∞.

Conversely, supppose that $(I-GK)$ is invertible and the transfer matrix in (10) is in RH_∞. Then, in particular, $(I-GK)^{-1}$ is proper which implies that $(I-GK)(\infty) = (I-D\hat{D})$ is invertible. Therefore

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}$$

is nonsingular. Now routine transfer function calculations give,

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \left[I + \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}(sI-\tilde{A})^{-1}\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Since the transfer matrix from (v_1,v_2) to (e_1,e_2) belongs to RH_∞, it follows that

$$\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}(sI-\tilde{A})^{-1}\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}$$

belongs to RH_∞. Finally, since (A,B,C) and $(\hat{A},\hat{B},\hat{C})$ are stabilizable and detectable, it follows that the eigenvalues of $\tilde{A}$ are in the open left half plane.

QED

We note that to check internal stability it is necessary (and sufficient) to check that each of the four transfer matrices in (11) are in RH_∞. It is not difficult to construct examples of G and K such that any three of the four transfer matrices in (11) are in RH_∞ while the fourth one is unstable.

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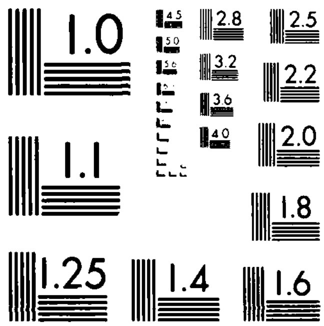

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2.2.3 Parametrization of All Stabilizing Controllers

Two matrices $N, M \in RH_\infty$ with the same number of columns are *right-coprime* if the combined matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ has a left inverse in RH_∞. That is, there exists $X, Y \in RH_\infty$ such that $XM + YN = I$. This is often called a Bezout or Diophantine equation. An alternative definition is that two matrices in RH_∞ are right-coprime if every common right divisor in RH_∞ is invertible in RH_∞. This can be shown to be equivalent to the above definition in terms of a left inverse, but we will not use this fact.

It is a fact that every $G \in R_p$ (proper, real-rational) has a right-coprime factorization $G = NM^{-1}$ where $N, M \in RH_\infty$ are right coprime. Similarly, there exist left coprime factorizations (lcf), defined in the obvious way by duality. The proof of the existence of such coprime factorizations will be given in the next section with explicit realizations for the factorizations. In this section, we will see how these factorizations can be used to obtain a parametrization of all stabilizing controllers.

Begin with rcf's and lcf's of G and K in Figure 4:

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{1}$$

$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U} \tag{2}$$

Lemma 1. Consider the system in Figure 4. The following conditions are equivalent:

1. The feedback system is internally stable.
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in RH_∞.
3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ is invertible in RH_∞.

Proof: As we saw in Lemma 2.3 of the last section, internal stability is

equivalent to the condition that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in RH_\infty$$

or, equivalently,

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1} \in RH_\infty. \tag{3}$$

Now

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix} = \begin{bmatrix} I & UV^{-1} \\ NM^{-1} & I \end{bmatrix}$$
$$= \begin{bmatrix} M & U \\ N & V \end{bmatrix}\begin{bmatrix} M^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$$

so that

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1} = \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1}.$$

Since the matrices

$$\begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}, \begin{bmatrix} M & U \\ N & V \end{bmatrix}$$

are right-coprime, (3) holds iff

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in RH_\infty.$$

This proves the equivalence of conditions 1 and 2. The equivalence of 1 and 3 is proved similarly.

QED

We shall see in the next section how to find explicit realizations for $N, M, \tilde{N}, \tilde{M}, U_0, V_0, \tilde{U}_0, \tilde{V}_0$ and such that (1) holds and

$$\begin{bmatrix} \tilde{V}_o & -\tilde{U}_o \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_o \\ N & V_o \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{5}$$

The above lemma says that

$$K_o \triangleq U_o V_o^{-1} = \tilde{V}_o^{-1} \tilde{U}_o$$

then qualifies a particular controller achieving internal stability. All stabilizing controllers can be expressed in terms of K_o and a parameter Q, as shown in the following:

Theorem 1. The set of all proper controllers achieving internal stability is parametrized by the formula

$$K = K_o + \tilde{V}_o^{-1} Q (I + V_o^{-1} N Q)^{-1} V_o^{-1} \tag{6}$$

where Q ranges over RH_∞ such that $(I + V_o^{-1} N Q)(\infty)$ is invertible.

Proof: Assume K has the form indicated.

Define

$$U \triangleq U_o + MQ, \quad V \triangleq V_o + NQ$$
$$\tilde{U} \triangleq \tilde{U}_o + Q\tilde{M}, \quad \tilde{V} \triangleq \tilde{V}_o + Q\tilde{N}$$

then

$$\begin{aligned} \begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} &= \begin{bmatrix} \tilde{V}_o + Q\tilde{N} & -(\tilde{U}_o + Q\tilde{M}) \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_o + MQ \\ N & V_o + NQ \end{bmatrix} \\ &= \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}_o & -\tilde{U}_o \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_o \\ N & V_o \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \qquad \text{from (5)} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad (7) \end{aligned}$$

Thus K achieves internal stability by lemma 1.

Conversely, suppose K is proper and it achieves internal stability. Introduce rcf and lcf of K as in (2).

Then by lemma 1, $Z \triangleq \tilde{M}V - \tilde{N}U$ is invertible in RH_∞. Define Q by the equation

$$U_0 + MQ = UZ^{-1}, \tag{8}$$

so

$$Q = M^{-1}(UZ^{-1} - U_0) \tag{9}$$

Then

$$\begin{aligned} V_0 + NQ &= V_0 + NM^{-1}(UZ^{-1} - U_0) \\ &= V_0 + \tilde{M}^{-1}\tilde{N}(UZ^{-1} - U_0) && \text{from (1)} \\ &= \tilde{M}^{-1}(\tilde{M}V_0 - \tilde{N}U_0 + \tilde{N}UZ^{-1}) \\ &= \tilde{M}^{-1}(I + \tilde{N}UZ^{-1}) && \text{from (5)} \\ &= \tilde{M}^{-1}(Z + \tilde{N}U)Z^{-1} \\ &= \tilde{M}^{-1}\tilde{M}VZ^{-1} \\ &= VZ^{-1} && (10) \end{aligned}$$

Thus,

$$\begin{aligned} K &= UV^{-1} \\ &= (U_0 + MQ)(V_0 + NQ)^{-1} \\ &= U_0 V_0 + (M - U_0 V_0^{-1} N)Q(I + V_0^{-1}NQ)^{-1}V_0^{-1} && (11) \end{aligned}$$

from (prelim?). Then, since

$$(M - U_0 V_0^{-1} N) = (M - \tilde{V}_0^{-1}\tilde{U}_0 N) = \tilde{V}_0^{-1}(\tilde{V}_0 M - \tilde{U}_0 N) = \tilde{V}_0^{-1}$$

we have that

$$K = U_e V_e + \tilde{V}_e^{-1} Q (I + V_e^{-1} N Q)^{-1} V_e^{-1}. \tag{12}$$

To see that Q belongs to RH_∞, observe first from (9) and (10) that MQ and NQ both do. Right-coprimeness of N and M then implies that $Q \in RH_\infty$.

Finally, since V and Z evaluated at $s=\infty$ are both invertible, so is $V_e + NQ$, from (10), hence so is $I + V_e^{-1} NQ$.

QED

Define the rational matrix

$$J \triangleq \begin{bmatrix} K_e & \tilde{V}_e^{-1} \\ V_e^{-1} & -V_e^{-1} N \end{bmatrix} \tag{13}$$

and consider a controller K given by formula (6). Then the controller equation

$$\begin{aligned} u &= \mathbf{F}_l(J, Q) y \\ &= \left[K_e + \tilde{V}_e^{-1} Q (I + V_e^{-1} N Q)^{-1} V_e^{-1}\right] y \end{aligned}$$

is equivalent to the triple of equations

$$\begin{aligned} u &= K_e y + \tilde{V}_e^{-1} y_1 \\ u_1 &= V_e^{-1} y - V_e^{-1} N y_1 \\ y_1 &= Q u_1 \end{aligned}$$

The block diagram corresponding to this triple is in Figure 5. We conclude that every stabilizing controller can be represented as $K = \mathbf{F}_l(J, Q)$, as in Figure 5, for some parameter Q, which is constrained only to be stable and proper and to make K proper.

The next section gives an explicit state-space realization of one choice of the interconnection matrix J.

2.2.4 **Realization of J :**

Recall from Section 2 that we have

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

where (A,B) is stabilizable and (C,A) is detectable. To obtain a right-coprime factorization of G, choose a matrix F such that $A+BF$ is stable.

Lemma 1. A stabilizing state feedback F yields rcf $G = NM^{-1}$ where

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right]. \qquad (1)$$

Proof: That $G = NM^{-1}$ follows from:

$$GM = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A & BF & B \\ 0 & A+BF & B \\ \hline C & DF & D \end{array}\right] \qquad \text{(cascade of two systems)}$$

$$= \left[\begin{array}{cc|c} A+BF & BF & B \\ 0 & A & 0 \\ \hline C+DF & DF & D \end{array}\right] \qquad \text{(by change of basis in the state-space)}$$

$$= \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right] \qquad \text{(deletion of uncontrollable part)}$$

$$= N.$$

That N and M are right-coprime will follow from (3) below.

QED

Note that for any nonsingular Z

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} A+BF & BZ \\ \hline F & Z \\ C+DF & DZ \end{array}\right]. \qquad (1a)$$

is also a realization of an rcf of G.

By duality, to get a left-coprime factorization of G, take H such that $A+HC$ is stable.

Lemma 1'. A stabilizing output injection H yields lcf $G=\tilde{M}^{-1}\tilde{N}$ where

$$\begin{bmatrix}\tilde{M} & \tilde{N}\end{bmatrix} := \left[\begin{array}{c|cc} A+HC & H & B+HD \\ \hline C & I & D \end{array}\right]. \tag{2}$$

The next step is to specify U_0, V_0, $\tilde{U}_0$, $\tilde{V}_0$ to satisfy

$$\begin{bmatrix}\tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M}\end{bmatrix}\begin{bmatrix}M & U_0 \\ N & V_0\end{bmatrix} = \begin{bmatrix}I & 0 \\ 0 & I\end{bmatrix}. \tag{3}$$

The idea behind the choice of these matrices is as follows. Using observer theory, find a controller K_0 achieving internal stability. Perform factorizations

$$K_0 = U_0 V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$$

analogous to the ones just performed on G. Then Lemma 3.1 implies that the left-hand side of (3) must be invertible in RH_∞. We shall see that, in fact, (3) is satisfied.

The equations for K_0 are

$$\dot{\hat{x}} = A\hat{x} + Bu + H(C\hat{x} + Du - y)$$
$$u = F\hat{x},$$

that is,

$$K_0 := \left[\begin{array}{c|c} A + BF + HC + HDF & -H \\ \hline F & 0 \end{array}\right]. \tag{4}$$

Define

$$\hat{A} := A + BF + HC + HDF \;,\; \hat{B} := -H$$

$$\hat{C} := F \;,\; \hat{D} := 0$$

$$\hat{F} := C + DF \;,\; \hat{H} := -(B + HD).$$

Following (1) and (2), define

$$\begin{bmatrix} V_0 \\ U_0 \end{bmatrix} := \left[\begin{array}{c|c} \hat{A} + \hat{B}\hat{F} & \hat{B} \\ \hline \hat{F} & I \\ \hat{C} + \hat{D}\hat{F} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} A+BF & -H \\ \hline C+DF & I \\ F & 0 \end{array}\right] \tag{5}$$

$$\begin{bmatrix} \tilde{V}_0 & \tilde{U}_0 \end{bmatrix} := \left[\begin{array}{c|cc} \hat{A} + \hat{H}\hat{C} & \hat{H} & \hat{B} + \hat{H}\hat{D} \\ \hline \hat{C} & I & \hat{D} \end{array}\right] = \left[\begin{array}{c|cc} A+HC & -(B+HD) & -H \\ \hline F & I & 0 \end{array}\right]. \tag{6}$$

Using the above definitions we have that

$$\begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right]. \tag{8}$$

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -(B+HD) & H \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]. \tag{28c}$$

and the following theorem holds.

Theorem 2.

Equation (3) is satisfied.

Proof: Verification of (3) is immediate using (7), (8), and the inversion formula for systems (prelim).

A realization of J is now immediate. Substitution of (1), (4), (5), and (6) into (3.13) leads after simplification to

$$J = \left[\begin{array}{c|cc} A+BF+HC+HDF & -H & B+HD \\ \hline F & 0 & I \\ -(C+DF) & I & -D \end{array}\right]. \tag{9}$$

Let's recap. We began with the (stabilizable and detectable) realization

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

We chose F and H so that $A+BF$ and $A+HC$ were stable. Define J by (9). Then the proper K's achieving internal stability are precisely those representable as in Figure 5, where $Q \in RH_\infty$ and

$$I + DQ(\infty) \text{ is invertible}$$

(The last condition is equivalent to the one

$$(I + V_0^{-1}NQ)(\infty) \text{ is invertible}$$

which is required as per Theorem 1).

This representation result has an interesting interpretation : every internal stabilization amounts to adding stable dynamics to the plant and then stabilizing the extended plant by means of an observer. The precise statement is as follows; for simplicity of the formulas, only the case of strictly proper G and K is treated.

Theorem 2.

Assume G and K are strictly proper and the system in Figure 3 is internally stable. Then G can be embedded in a system

$$\left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & 0 \end{array}\right],$$

where

$$A_e = \begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \end{bmatrix} \tag{10}$$

and A_a is stable, such that K has the form

$$K = \left[\begin{array}{c|c} A_e + B_e F_e + H_e C_e & -H_e \\ \hline F_e & 0 \end{array}\right], \tag{11}$$

where $A_e + B_e F_e$ and $A_e + H_e C_e$ are stable.

Proof. K is representable as in Figure 5 for some Q in RH_∞. For K to be strictly proper, so must Q be (see (3.6)). Take a minimal realization of Q:

$$Q = \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & 0 \end{array}\right].$$

Since $Q \in RH_\infty$, A_a is stable. Let x and x_a denote state vectors for J and Q respectively, and write the equations for the system in Figure 5 :

$$\dot{x} = (A + BF + HC)x - Hy + By_1$$

$$u = Fx + y_1$$

$$u_1 = -Cx + y$$

$$\dot{x}_a = A_a x_a + B_a u_1$$

$$y_1 = C_a x_a$$

These equations yield

$$\dot{x}_e = (A_e + B_e F_e - H_e C_e)x_e - H_e y$$

$$u = F_e x_e ,$$

where

$$x_e := \begin{bmatrix} x \\ x_a \end{bmatrix}, \quad F_e := \begin{bmatrix} F & C_a \end{bmatrix}, \quad H_e := \begin{bmatrix} H \\ -B_a \end{bmatrix}$$

and A_e, B_e, C_e are as in (10).

QED

2.2.5 Closed-Loop Transfer Matrix

Theorem 1 provides a parametrization, in terms of Q, of all proper K's which achieve internal stability in Figure 1. The goal in this section is to express the transfer matrix from v to e in terms of Q.

A stabilizing K is representable as in Figure 5. Substitution of the block diagram in Figure 5 into that in Figure 1 leads to the one in Figure 6. Elimination of the signals u and y leads to Figure 7 for a suitable transfer matrix T. Thus all closed-loop transfer matrices are representable as in Figure 7. It remains to give a realization of T.

We must first put back the original notation which was simplified at the end of Section 2. Let

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1}$$

be a minimal realization of P, and choose F and H so that $A+B_2F$ and $A+HC_2$ are stable.

Lemma 4.

$$T = \left[\begin{array}{cc|cc} A+B_2F & -B_2F & B_1 & B_2 \\ 0 & A+HC_2 & B_1+HD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right] \tag{2}$$

Proof. With the original notation we have from (..4.9) that

$$J = \left[\begin{array}{c|cc} A+B_2F+HC_2+HD_{22}F & -H & B_2+HD_{22} \\ \hline F & 0 & I \\ -(C_2+D_{22}F) & I & -D_{22} \end{array}\right] \tag{3}$$

Partition J and T accordingly:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

The equations governing the system in Figure 6 are

$$\begin{bmatrix} e \\ u_1 \end{bmatrix} = \begin{bmatrix} P_{11} & 0 \\ 0 & J_{22} \end{bmatrix}\begin{bmatrix} v \\ y_1 \end{bmatrix} + \begin{bmatrix} P_{12} & 0 \\ 0 & J_{21} \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}$$

$$\begin{bmatrix} I & -J_{11} \\ -P_{22} & I \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & J_{12} \\ P_{21} & 0 \end{bmatrix}\begin{bmatrix} v \\ y_1 \end{bmatrix},$$

while those for Figure 7 are

$$\begin{bmatrix} e \\ u_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} v \\ y_1 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & 0 \\ 0 & J_{22} \end{bmatrix} + \begin{bmatrix} P_{12} & 0 \\ 0 & J_{21} \end{bmatrix}\begin{bmatrix} I & -J_{11} \\ -P_{22} & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & J_{12} \\ P_{21} & 0 \end{bmatrix}. \tag{4}$$

The strategy now is to use the representations (1) and (3) in (4) to obtain (2). The algebra is long and tedious, but straightforward, so the details are omitted.

Partition T according to (2):

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

Explicitly, we have

$$T_{11} = \left[\begin{array}{cc|c} A+B_2F & -B_2F & B_1 \\ 0 & A+HC_2 & B_1+HD_{21} \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} \end{array}\right] \tag{5a}$$

state feedback. Using the notation

$$N = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} A+BF & BZ \\ \hline C+DF & DZ \end{array}\right] \tag{8}$$

we will use (1)-(3) to get N inner and $A+BF$ stable. From (2) we have that $Z=R^{-1/2}U$ where $R=D'D>0$ and U is any orthogonal matrix. Take $U=I$. Equation (1) implies that

$$R^{-1/2}B'X + R^{-1/2}D'(C+DF) = 0$$

so solving for F yields

$$F = -R^{-1}(B'X + D'C) \tag{9}$$

Then equation (3) yields

$$\begin{aligned} 0 &= \hat{A}'X + X\hat{A}' - \hat{C}'\hat{C} \\ &= (A+BF)'X + X(A+BF) + (C+DF)'(C+DF) \\ &= (A-BR^{-1}D'C-BR^{-1}B'X)'X + X(A-BR^{-1}D'C-BR^{-1}B'X) \\ &\qquad + (C-DR^{-1}B'X-DR^{-1}D'C)'(C-DR^{-1}B'X-DR^{-1}D'C) \\ &= (A-BR^{-1}D'C)'X + X(A-BR^{-1}D'C) - XBR^{-1}B'X + C'D_\perp D_\perp' C \end{aligned} \tag{10}$$

since $D_\perp D_\perp' = I-DR^{-1}D'$. Thus $X=\mathrm{Ric}\,[A_H]$, where

$$A_H = \begin{bmatrix} A-BR^{-1}D'C & -BR^{-1}B' \\ -C'D_\perp D_\perp' C & -(A-BR^{-1}D'C)' \end{bmatrix} \tag{11}$$

That $X=\mathrm{Ric}\,(A_H)$ exists such that $A+BF$ is stable follows from Theorem 2.1 as follows. Let

$$\begin{aligned} \Gamma &= G^*G \\ &= \begin{bmatrix} B'(-sI-A')^{-1} & I \end{bmatrix} \begin{bmatrix} C'C & C'D \\ D'C & D'D \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} \end{aligned} \tag{12}$$

That $\Gamma = G^*G > 0$ is true by assumption. To satisfy Theorem 2.1 we must have (A, B) stabilizable and (P, A) detectable, but this is immediate since

and Z can be any nonsingular matrix. To obtain a rcf with N inner, we simply need to use equations (1)-(4) to solve for F and Z. This yields the following theorem:

Theorem 1 :

Assume $p \geq m$. Then, there exists a rcf $G = NM^{-1}$ with N inner if and only if $G^*G > 0$ on the $j\omega$-axis, including at ∞. This factorization is unique up to a constant unitary multiple.

A particular realization for the factorization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & BR^{-1/2} \\ \hline F & R^{-1/2} \\ C+DF & DR^{-1/2} \end{array}\right] \in RH_\infty^{(m+p)\times m} \tag{5}$$

where

$$R = D'D > 0$$
$$F = -R^{-1}(B'X + D'C) \tag{6}$$

and

$$X = \mathrm{Ric}\begin{bmatrix} A-BR^{-1}D'C & -BR^{-1}B' \\ -C'D_\perp D_\perp' C & -(A-BR^{-1}D'C)' \end{bmatrix} \geq 0 \tag{7}$$

[Proof] :

(only if) :

Suppose $G = NM^{-1}$ is a rcf and $N^*N = I$. Then $G^*G = (NM^{-1})^*(NM^{-1}) = (M^{-1})^*M^{-1} > 0$ on the $j\omega$-axis since $M \in RH_\infty$.

(if) :

The if part will be proven by showing that (1)-(4) lead directly to the above realization of the rcf of G with inner numerator. That $G = NM^{-1}$ is an rcf follows immediately from (4) once it is established that F is a stabilizing

2.3.4 Inner-Outer and Spectral Factorization :

In this section, the special form of coprime factorizations required to reduce the general H_∞ optimal control problem to a best approximation problem will be developed. In particular, explicit realizations are given for coprime factorizations $G = NM^{-1}$ with inner numerator N (Theorem 1) and inner denominator M (Theorem 3); and for the complementary inner factor $N_\perp$ which completes the inner numerator to make $\begin{bmatrix} N & N_\perp \end{bmatrix}$ square and inner (Theorem 2). The theorems will be stated for right coprime factorizations (rcf) with the duals for lcf's following just as for the general case of coprime factorization developed earlier.

For the following theorems, it is assumed that $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{R}_p^{p\times m}$ and the realization is minimal. We will denote by $R^{1/2}\,(R\geq 0)$ the symmetric matrix such that $R^{1/2}R^{1/2} = R$ and use "$D_\perp$" for any orthogonal complement of D so that $\begin{bmatrix} DR^{-1/2} & D_\perp \end{bmatrix}$ (with $R = D'D$) is square and orthogonal.

Recall from Corollary 0.3.3.1 that $N = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ is inner if and only if

i) $\hat{B}'X + \hat{D}'\hat{C} = 0$ (1)

ii) $\hat{D}'\hat{D} = I$ (2)

where the observability gramian X solves

$$\hat{A}'X + X\hat{A} + \hat{C}'\hat{C} = 0 \tag{3}$$

From Lemma 2.4.1 a stabilizing state feedback F yields rcf $G = NM^{-1}$ where

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & BZ \\ \hline F & Z \\ C+DF & DZ \end{array}\right]. \tag{4}$$

Remark :

The unique stabilizing solution of Theorem 1 will be denoted by $\mathrm{Ric}\,(A_H)$. Note that this theorem is more general than Theorem 2.1 from the previous section since no detectability assumptions are made. The following theorem will play an important role in the next section in obtaining complementary inner factors.

Theorem 2 :

If $Q = H^T H \geq 0$ in (ARE) and X is its solution, then $\mathbf{Ker}(X) \subset \mathbf{Ker}(H)$.

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$$

$$= \begin{bmatrix} E-WX & -W \\ -(E^TX+XE-XWX+Q) & -(E-WX)^T \end{bmatrix}$$

$$= \begin{bmatrix} E-WX & -W \\ 0 & -(E-WX)^T \end{bmatrix}.$$

This puts A_H in block upper triangular form and clearly exhibits a particular partitioning of the eigenvalues of A_H with respect to the imaginary axis. For example, if $E-WX$ has all its eigenvalues in $\mathbb{C}_-$, then $-(E-WX)^T$ has all its poles in $\mathbb{C}_+$. Thus, the solution of ARE which stabilizes $E-WX$ yield a decomposition of A_H into stable and unstable parts.

This section will explore the conditions under which the desired solution of ARE exists. There is a considerable literature addressing the theory of ARE (eg. [And], [Cop], [Kuc], [Mat], [*Mol*], [Pot], [Wil]), and it is not the purpose of these notes to give a detailed treatment of this subject. We will simply review the results which are relevant to the factorization theorems in this report.

Now, we are going to state the main theorem of this section which gives the necessary and sufficient conditions for the existence of a unique stabilizing solution of (ARE). Without loss of generality, we will assume that $W = GG^T$.

Theorem 1 :

The stabilizability of (E,G) and $\mathrm{Re}[\lambda_i(A_H)] \neq 0$ ($\forall \;\; i = 1,2,\ldots\ldots,2n$) is necessary as well as sufficient for the existence of a unique stabilizing solution of (ARE).

2.3.3 Solution of the Algebraic Riccati Equation :

Consider once again the Algebraic Riccati Equation,

$$E^T X + XE - XWX + Q = 0 \tag{ARE}$$

where

$$E,\ W,\ Q \ \in \ \mathbb{R}^{n\times n},\ \ W = W^T \geq 0 \ \text{ and } \ Q = Q^T$$

with the associated Hamiltonian matrix

$$A_H = \begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix} \tag{Hamiltonian}$$

Our main interest is to find the unique real symmetric stabilizing solution such that the matrix $(E - WX)$ is asymptotically stable. For simplicity we will use "solution" of the ARE to mean a real symmetric one. The ARE considered here is more general than the ARE which arises in linear quadratic optimal control and Kalman-Bucy filtering theory in that there is no assumption on the definiteness of the matrix Q.

An important property of the Hamiltonian matrix A_H is that the distribution of its eigenvalues (denoted as $\Lambda(A_H)$) is symmetric with respect to both the real and imaginary axes, i.e., if $\lambda \in \Lambda(A_H)$ with multiplicity k, so is $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$. Therefore, Λ can be partitioned as Λ_1 and Λ_2 so that $\lambda \in \Lambda_1$ with multiplicity k implies that $\bar{\lambda} \in \Lambda_1$ and $-\lambda, -\bar{\lambda} \in \Lambda_2$ all with the same multiplicity.

One connection between the ARE and A_H can be seen by assuming that X is a solution to ARE and conjugating A_H by $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ to yield

The next section focuses on the solution of the Riccati equation.

then from (4) and (5)

$$(j\omega_0 I - A)x_1 = 0 \tag{7}$$

$$(j\omega_0 I + A')x_2 = -Px_1 \tag{8}$$

Since (7) implies $x_1^*(j\omega_0 I + A') = 0$, from (8) we have $x_1^* P x_1 = 0$. This implies, along with (7) that (P,A) is not detectable. Hence $\hat{C}x_0 \neq 0$. Now Lemma 0.2.3.8 implies that there exists $u_0 \neq 0$ such that $\Gamma(j\omega)u_0 = 0$. This contradicts the hypothesis that $\Gamma(j\omega) > 0$. Hence (a) $\rightarrow$ (c).

(b) $\rightarrow$ (a) Suppose $\exists X = X'$ such that $E - BR^{-1}B'X = A - BR^{-1}(S' + B'X)$ is stable. Let $F = -R^{-1}(S' + B'X)$ and

$$M = \left[\begin{array}{c|c} A & B \\ \hline -F & I \end{array}\right]$$

It is easily verified by use of the Riccati equation for X and routine algebra that $\Gamma = M^*RM$ so

$$\Gamma^{-1}(s) = M^{-1}(s)R^{-1}(M'(-s))^{-1}$$

Now

$$M^{-1} = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right]$$

So $M^{-1} \in RH_\infty$. Thus $\Gamma^{-1} \in RL_\infty$ and for all $0 \leq \omega \leq \infty$

$$\Gamma^{-1}(j\omega) = M^{-1}(j\omega)R(M'(-j\omega))^{-1} > 0$$

Hence $\Gamma(j\omega) > 0$ and (b) $\rightarrow$ (a).

(c) $\rightarrow$ (b) This is part of Theorem 3.1 in the next section.

QED

c) The Hamiltonian matrix

$$A_H = \begin{bmatrix} A-BR^{-1}S' & -BR^{-1}B' \\ -P+SR^{-1}S' & -(A-BR^{-1}S')' \end{bmatrix}$$

has no $j\omega$-axis eigenvalues.

Corollary 1 If the conditions in Theorem 1 are satisfied then $\exists M \in R_p$ such that $M^{-1} \in RH_\infty$ and

$$\Gamma = M^* R M.$$

A particular realization of one such M is

$$M = \left[\begin{array}{c|c} A & B \\ \hline -F & I \end{array}\right]$$

where $F = -R^{-1}(S' + B'X)$.

Proof: (a)→(c) Let

$$\Gamma(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \triangleq \left[\begin{array}{cc|c} A & 0 & B \\ -P & -A' & -S \\ \hline S' & B' & R \end{array}\right] \tag{2}$$

Then $A_H = \hat{A} - \hat{B}\hat{D}^{-1}\hat{C}$. Suppose A_H has an eigenvalue on the $j\omega$-axis. Then $\exists \omega_o, x_o = (x_1', x_2')'$ such that

$$A_H x_o = j\omega_o x_o \tag{3}$$

or

$$(j\omega_o I - A)x_1 = -BR^{-1}(S'x_1 + B'x_2) \tag{4}$$

$$(j\omega_o I + A')x_2 = -(P - SR^{-1}S')x_1 + SR^{-1}B'x_2 \tag{5}$$

Suppose

$$0 = \hat{C}x_o = S'x_1 + B'x_2 \tag{6}$$

2.3.2 Riccati Equations and Factorizations

Consider the Algebraic Riccati Equation,

$$E'X + XE - XWX + Q = 0 \qquad \text{(ARE)}$$

where

$$E,\ W,\ Q \ \in \ \mathbb{R}^{n\times n},\ \ W = W' \geq 0 \ \text{ and } \ Q = Q'$$

with the associated Hamiltonian matrix

$$A_H = \begin{bmatrix} E & -W \\ -Q & -E' \end{bmatrix} \qquad \text{(Hamiltonian)}$$

The following theorem and corollary characterizes the relationship between spectral factorization, Riccati equations, and decomposition of Hamiltonians.

Theorem 1 Let A,B,P,S,R be matrices of compatible dimensions such that $P=P'$, $R=R'>0$, with (A,B) stabilizable and (P,A) detectable. Then the following statements are equivalent.

a) The parahermitian rational matrix

$$\Gamma(s) \ = \ \begin{bmatrix} B'(-sI-A')^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$$

satisfies

$$\Gamma(j\omega) > 0 \quad \text{for all} \quad 0 \leq \omega \leq \infty$$

b) For $E=A-BR^{-1}S'$, $W=BR^{-1}B'$ and $Q=P-SR^{-1}S'$, there exists a unique real $X=X'$ such that

$$E'X + XE - XWX + Q = 0$$

and $E-BR^{-1}B'X$ is stable.

The key idea behind the factorizations in this chapter is the connection between inner functions, Riccati equations, and spectral factorization. Indeed, the factorization $(\gamma^2 I - G^* G)^{1/2}$ is just a special case of spectral factorization ([You],[And]). Sections 2 and 3 develop the fundamental properties about Riccati equations that will be needed in Section 4 to construct the desired factorizations. The material in Sections 2 and 3 is for the most part well-known within the control theory community ([And],[Cop],[Kuc],[Mat],[Mol],[Pot],[Wil]), although Theorem 1 of Section 2 is apparently a somewhat novel description of the connection between spectral factorization, Riccati equations, and decomposition of Hamiltonians.

Section 4 shows that coprime factorizations with inner numerator and complementary inner factors both involve using a state feedback or output *injection matrix based on* a Riccati solution. This provides a reliable computational method based on standard approaches to finding solutions of Riccati equations ([Pot],[Lau]). Section 5 completes the solution to the H_∞ optimal control problem by completing the parametrization of the optimal controller using the factorizations developed in Section 4.

2.3 Factorization

2.3.1 Introduction to Factorization

The last chapter developed methods for finding J's so that the substitution $K=\mathbf{F}_l(J,Q)$ yields

$$\begin{aligned}\mathbf{F}_l(P,K) &= \mathbf{F}_l(P,\mathbf{F}_l(J,Q))\\ &= \mathbf{F}_l(T,Q)\\ &= T_{11} + T_{12}QT_{21} \qquad (1)\end{aligned}$$

with the additional requirement that $T \in H_\infty$ and

$$\mathbf{F}_l(P,K) \quad \text{internally stable} \qquad (2)$$
$$\text{iff} \quad Q \in H_\infty.$$

This parametrizes all stabilizing K's in terms of a stable $Q \in H_\infty$ in addition to providing an affine parametrization of all stable $\mathbf{F}_l(P,K)$. The actual structure of stabilizing parametrizations J was in terms of an observer-based compensator. The stabilizing state feedback and output injections of the observer-based compensator were shown to provide coprime factorizations of P and solve the Bezout identities necessary to provide the parametrization of all stabilizing controllers.

In this chapter, the requirement is added that T_{12} and T_{21} be inner, that is $T_{12}^*T_{12}=I$ and $T_{21}T_{21}^*=I$. In addition, we find $T_\perp$ and $\tilde{T}_\perp$ so that $\begin{bmatrix}T_{12} & T_\perp\end{bmatrix}$ and $\begin{bmatrix}T_{21}\\ \tilde{T}_\perp\end{bmatrix}$ are *square* and inner. This provides the necessary factorizations to complete step 2) in Section 2.1.5 on the Rational Matrix Generalization. The final missing step in completing the solution in the rational case is to find $M \in RH_\infty$ for a given $G \in RL_\infty$ with $\|G\|_\infty < \gamma$ such that $M^{-1} \in RH_\infty$ and $M^*M=(\gamma^2 I-G^*G)$. The symbol $(\gamma^2 I-G^*G)^{1/2}$ is used to denote this M.

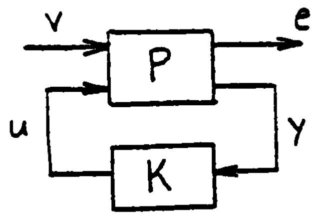

Figure 1

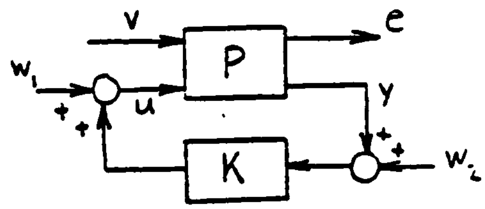

Figure 1a

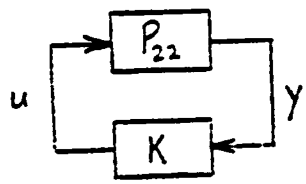

Figure 2

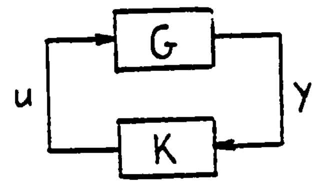

Figure 3

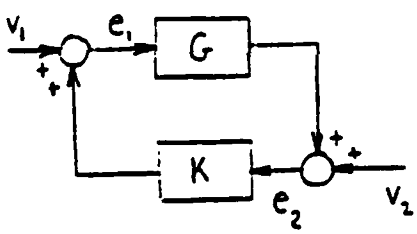

Figure 4

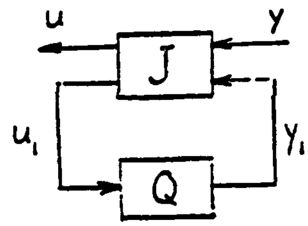

Figure 5

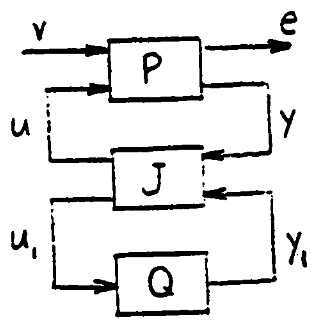

Figure 6

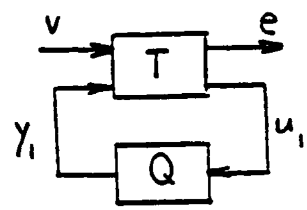

Figure 7

$$T_{12} = \left[\begin{array}{c|c} A+B_2F & B_2 \\ \hline C_1+D_{12}F & D_{12} \end{array}\right] \tag{5b}$$

$$T_{21} = \left[\begin{array}{c|c} A+HC_2 & B_1+HD_{21} \\ \hline C_2 & D_{21} \end{array}\right] \tag{5c}$$

$$T_{22} = 0.$$

In Figure 7 the governing equations are therefore

$$\begin{aligned} z &= T_{11}v + T_{12}y_1 \\ u_1 &= T_{21}v \\ y_1 &= Qu_1. \end{aligned}$$

so that

$$z = (T_{11} + T_{12}QT_{21})v.$$

In summary, we have

Theorem 3. The set of all closed-loop transfer matrices from v to z achievable by an internally stabilizing proper controller is equal to

$$\left\{T_{11} + T_{12}QT_{21} : Q \in RH_\infty, \quad I + D_{22}Q(\infty) \text{ invertible}\right\}.$$

The important points to note are that the closed-loop transfer matrix is simply an affine function of the controller parameter matrix Q and that the coefficient matrices T_{ij} have very simple realizations, namely, as in (5).

the realization for G was assumed minimal. Thus, Theorem 2.1 ensures that $X = \mathrm{Ric}\,(A_H)$ exists such that $A+BF$ is stable.

The uniqueness of the factorization follows from coprimeness and N inner. Suppose that $G = N_1M_1^{-1} = N_2M_2^{-1}$ are two right coprime factorizations and that both numerators are inner. By coprimeness, these two factorizations are unique up to a right multiple which is a unit in $RH_\infty^{m\times m}$. That is, there exists a unit $\Theta \in RH_\infty^{m\times m}$, such that $\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \Theta = \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}$. Clearly, Θ is inner since $\Theta^*\Theta = \Theta^* N_1^* N_1 \Theta = N_2^* N_2 = I$. The only inner units in RH_∞ are constant matrices, and thus the desired uniqueness property is established. Note that the nonuniqueness is contained entirely in the choice of a particular square root of R.

Q.E.D.

In a similar manner equations (1)-(3) can be used to obtain the complementary inner factor (CIF) in the following theorem.

Theorem 2 :

If $p > m$ in Theorem 1, then there exists a CIF $N_\perp \in RH_\infty^{p\times(p-m)}$ such that the matrix $\begin{bmatrix} N & N_\perp \end{bmatrix}$ is square and inner. A particular realization is $N_\perp = \left[\begin{array}{c|c} A+BF & -X^\dagger C' D_\perp \\ \hline C+DF & D_\perp \end{array}\right]$ where X and F are from Theorem 1 and $X^\dagger$ is the pseudo-inverse of X .

[Proof] :

The proof consists of verifying directly that $\begin{bmatrix} N & N_\perp \end{bmatrix}$ is inner using the above realization for $N_\perp$ and the realization for N from Theorem 1. Using the notation

$$\begin{bmatrix} N & N_\perp \end{bmatrix} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|cc} A+BF & BR^{-\frac{1}{2}} & -X^{\dagger}C'D_\perp \\ \hline C+DF & DR^{-\frac{1}{2}} & D_\perp \end{array}\right] \tag{13}$$

and the fact that $\mathrm{Ker}(X) \subset \mathrm{Ker}(D_\perp' C)$ (Theorem 3.2), equations (1)-(3) follow immediately. Thus $\begin{bmatrix} N & N_\perp \end{bmatrix}$ is inner.

Theorem 3 :

There exists a rcf $G = NM^{-1}$ such that $M \in RH_\infty^{m \times m}$ is inner if and only if G has no poles on the $j\omega$-axis. A particular realization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right] \in RH_\infty^{(m+p) \times m} \tag{14}$$

where

$$F = -B'X \tag{15}$$

and

$$X = \mathrm{Ric} \begin{bmatrix} A & -BB' \\ 0 & -A' \end{bmatrix} \geq 0 \tag{16}$$

[Proof] :

The proof is essentially the same as for Theorem 1. The details are straightforward and are omitted.

In the following theorem, we may assume that $G(s)$ is stable without loss of generality. Any $G \in RL_\infty$ may be factored using the dual of Theorem 3 to obtain a stable numerator $\tilde{N}$ such that $\tilde{N}^*\tilde{N} = G^*G$.

Theorem 4 : (Spectral Factorization)

Assume $G(s) \in RH_\infty^{p\times m}$ and $\gamma > ||G(s)||_\infty$. Then, there exists a $M \in RH_\infty^{m\times m}$ with stable inverse such that $M^*M = \gamma^2 I - G^*G$ with

$$M = \left[\begin{array}{c|c} A & B \\ \hline -R^{\frac{1}{2}}K_C & R^{\frac{1}{2}} \end{array}\right]$$

where

$$R = \gamma^2 I - D'D > 0$$

$$K_C = -R^{-1}(B'X - D'C)$$

$$X = \text{Ric}\begin{bmatrix} A+BR^{-1}D'C & -BR^{-1}B' \\ C'(I+DR^{-1}D')C & -(A+BR^{-1}D'C)' \end{bmatrix}$$

[Proof] :

Let

$$\Gamma = \gamma^2 I - G^*G = \begin{bmatrix} B'(-sI-A')^{-1} & I \end{bmatrix}\begin{bmatrix} -C'C & -C'D \\ -D'C & R \end{bmatrix}\begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$$

where $R = \gamma^2 I - D'D$.

Since $\gamma > ||G||_\infty$, $\Gamma(j\omega)>0$. The minimality of the realization of $G(s)$ guarantees that (A, B) is controllable and $(-C'C, A)$ is observable. Thus, from Corollary 2.1, there exists $M(s) \in R_p^{m\times m}$ such that $\Gamma = M^*M$ and a particular realization is

$$M = \left[\begin{array}{c|c} A & B \\ \hline -R^{\frac{1}{2}}K_c & R^{\frac{1}{2}} \end{array}\right]$$

where

$$K_c = -R^{-1}(B'X - D'C)$$

and

$$X = \mathrm{Ric}\begin{bmatrix} A+BR^{-1}D'C & -BR^{-1}B' \\ C'(I+DR^{-1}D')C & -(A+BR^{-1}D'C)' \end{bmatrix}.$$

Since G is stable, we conclude that $M \in RH_\infty^{m \times m}$.

Q.E.D.

Remarks :

(1) The minimality condition in Theorem 3 can be weakened to (A,B) stabilizable and A has no eigenvalues on the $j\omega$-axis and the theorem still holds.

(2) If $G \in RH_\infty^{p \times m}$ in Theorem 1, then M is a unit in RH_∞ and M^{-1} is "outer". In this case, $G = N(M^{-1})$ is called "inner-outer factorization" (IOF).

(3) Dual results for all factorizations can be obtained when $p \leq m$. In these factorizations, output injection using the dual Riccati solution replaces state feedback to obtain corresponding left factorizations.

(4) The symbol $(\gamma^2 I - G^* G)^{1/2}$ will be used to denote the M in Theorem 4.

2.3.5 Parametrizing the Optimal Controller

This section combines the results of Youla's parametrization and the coprime factorization to parameterize all stabilizing controllers in a way that is convenient for solving optimal L_2 and L_∞ control problems. This completes the rational generalization of the constant matrix case as outlined in Section 2.1.5. The only remaining step is a standard H_∞ approximation problem.

Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1}$$

Suppose that neither P_{12} nor P_{21} has transmission zeros on the $j\omega$-axis (including ∞) and, without loss of generality, $D_{12}^T D_{12} = I$ and $D_{21} D_{21}^T = I$. Under these assumptions, let $D_\perp = (D_{12})_\perp$ and $\tilde{D}_\perp = (D_{21})_\perp$ that is, $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ and $\begin{bmatrix} D_{21}^T & \tilde{D}_\perp^T \end{bmatrix}$ are orthogonal matrices. Then, factor P as before with F and H given as follows :

$$F = -(D_{12}^T C_1 + B_2^T X)$$

$$X = \mathrm{Ric} \begin{bmatrix} A - B_2 D_{12}^T C_1 & -B_2 B_2^T \\ -C_1^T D_\perp D_\perp^T C_1 & -(A - B_2 D_{12}^T C_1)^T \end{bmatrix} \tag{2}$$

and

$$H = -(B_1 D_{21}^T + Y C_2^T)$$

$$Y = \mathrm{Ric} \begin{bmatrix} (A - B_1 D_{21}^T C_2)^T & -C_2^T C_2 \\ -B_1 \tilde{D}_\perp^T \tilde{D}_\perp B_1^T & -(A - B_1 D_{21}^T C_2) \end{bmatrix} \tag{3}$$

Then, $N_{12}^* N_{12} = I$ and $\tilde{N}_{21} \tilde{N}_{21}^* = I$. Also, let $N_\perp$ and $\tilde{N}_\perp$ be CIF's so that

$$\begin{bmatrix} N_{12} & N_\perp \end{bmatrix} = \left[\begin{array}{c|cc} A + B_2 F & B_2 & -X^\dagger C_1^T D_\perp \\ \hline C_1 + D_{12} F & D_{12} & D_\perp \end{array}\right]$$

$$\begin{bmatrix} \tilde{N}_{21} \\ \tilde{N}_{\perp} \end{bmatrix} = \left[\begin{array}{c|c} A + HC_2 & B_1 + HD_{21} \\ \hline C_2 & D_{21} \\ -\tilde{D}_{\perp} B_1^T Y^{\dagger} & \tilde{D}_{\perp} \end{array}\right] \tag{4}$$

Letting

$$T = \begin{bmatrix} (N\tilde{V})_{11} & N_{12} \\ \tilde{N}_{21} & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right] \tag{5}$$

reduces

$$\min_{K \in R_p} \left\{ \left\| \mathbf{F}_l(P,K) \right\|_{\alpha} \,\middle|\, \mathbf{F}_l(P,K) \text{ stable} \right\} \tag{6}$$

to

$$\min_{Q \in RH_{\infty}} \left\{ \left\| \mathbf{F}_l(T,Q) \right\|_{\alpha} \right\}$$

$$= \min_{Q \in RH_{\infty}} \left\{ \left\| (N\tilde{V})_{11} + N_{12}Q\tilde{N}_{21} \right\|_{\alpha} \right\} \tag{7}$$

The optimal K may be recovered from Q.

Because both the $\|\cdot\|_2$ and $\|\cdot\|_{\infty}$ norms are unitary invariant, an alternative expression is possible. For any $Q \in RH_{\alpha}$ ($\alpha = 2, \infty$), we have

$$\left\| (N\tilde{V})_{11} + N_{12}Q\tilde{N}_{21} \right\|_{\alpha}$$

$$= \left\| \begin{bmatrix} N_{12} & N_{\perp} \end{bmatrix}^* \left[(N\tilde{V})_{11} + N_{12}Q\tilde{N}_{21} \right] \begin{bmatrix} \tilde{N}_{21} \\ \tilde{N}_{\perp} \end{bmatrix}^* \right\|_{\alpha}$$

$$= \left\| \begin{bmatrix} N_{12}^*(N\tilde{V})_{11}\tilde{N}_{21}^* + Q & N_{12}^*(N\tilde{V})_{11}\tilde{N}_{\perp}^* \\ N_{\perp}^*(N\tilde{V})_{11}\tilde{N}_{21}^* & N_{\perp}^*(N\tilde{V})_{11}\tilde{N}_{\perp}^* \end{bmatrix} \right\|_{\alpha}$$

$$= \left\| \begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\alpha} \tag{8}$$

$$\text{where } R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} N_{12}^* \\ N_\perp^* \end{bmatrix} (N\tilde{V})_{11} \begin{bmatrix} \tilde{N}_{21}^* & \tilde{N}_\perp^* \end{bmatrix}. \tag{9}$$

The $\alpha = 2$ case is particularly simple and it possible to get explicit formulas for the optimal controller. Since

$$\left\| \begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2 = \left(\left\| R_{11} + Q \right\|_2^2 + \left\| \begin{bmatrix} 0 & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_2^2 \right)^{1/2} \tag{10}$$

the optimal Q is seen immediately to be

$$Q_{opt} = \left\{ R_{11} \right\}_+ \tag{11}$$

To obtain an explicit expression for Q_{opt}, we need to compute R.

Note that $(N\tilde{V})_{11} = \begin{bmatrix} N_{11} & N_{12} \end{bmatrix} \begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix}$. It is convenient to compute

$$\begin{bmatrix} N_{12}^* \\ N_\perp^* \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \end{bmatrix} = \begin{bmatrix} N_{12}^* N_{11} & I \\ N_\perp^* & 0 \end{bmatrix} \tag{12}$$

and

$$\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix} \begin{bmatrix} \tilde{N}_{21}^* & \tilde{N}_\perp^* \end{bmatrix} \tag{13}$$

seperately.

Claim 1 :

$$\begin{bmatrix} N_{12}^* \\ N_\perp^* \end{bmatrix} \begin{bmatrix} N_{11} \end{bmatrix} = \left[\begin{array}{c|c} -(A + B_2F)^T & (C_1 + D_{12}F)^T D_{11} + XB_1 \\ \hline -B_2^T & D_{12}^T D_{11} \\ D_\perp^T C_1 X^+ & D_\perp^T D_{11} \end{array} \right] \tag{14}$$

[Proof] :

$$\begin{bmatrix} N_{12}^{*} \\ N_{\perp}^{*} \end{bmatrix} \begin{bmatrix} N_{11} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2F & 0 & B_1 \\ -(C_1 + D_{12}F)^T(C_1 + D_{12}F) & -(A + B_2F)^T & -(C_1 + D_{12}F)^T D_{11} \\ \hline D_{12}^T(C_1 + D_{12}F) & B_2^T & D_{12}^T D_{11} \\ D_{\perp}^T C_1 & -D_{\perp}^T C_1 X^{\dagger} & D_{\perp}^T D_{11} \end{array}\right]$$

conjugating the states by $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$, we have

$$\begin{bmatrix} N_{12}^{*} \\ N_{\perp}^{*} \end{bmatrix} \begin{bmatrix} N_{11} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2F & 0 & B_1 \\ 0 & -(A + B_2F)^T & -(C_1 + D_{12}F)^T D_{11} - XB_1 \\ \hline D_{12}^T(C_1 + D_{12}F) + B_2^T & B_2^T & D_{12}^T D_{11} \\ 0 & -D_{\perp}^T C_1 X^{\dagger} & D_{\perp}^T D_{11} \end{array}\right]$$

Since $D_{12}^T(C_1 + D_{12}F) + B_2^T X = D_{12}^T C_1 + B_2^T X + F = 0$, the claim is verified.

Q.E.D.

Claim 2:

$$\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix} \begin{bmatrix} \tilde{N}_{21}^{*} & \tilde{N}_{\perp}^{*} \end{bmatrix} = \left[\begin{array}{c|cc} -(A + HC_2)^T & -C_2^T & Y^{\dagger} B_1 \tilde{D}_{\perp}^T \\ \hline (B_1 + HD_{21})^T & D_{21}^T & \tilde{D}_{\perp}^T \\ FY & 0 & 0 \end{array}\right] \tag{15}$$

[Proof]:

$$\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix} \begin{bmatrix} \tilde{N}_{21}^{*} & \tilde{N}_{\perp}^{*} \end{bmatrix} = \left[\begin{array}{cc|cc} A + HC_2 & -(B_1 + HD_{21})^T(B_1 + HD_{21}) & (B_1 + HD_{21})D_{21}^T & B_1 \tilde{D}_{\perp}^T \\ 0 & -(A + HC_2)^T & C_2^T & -Y^{\dagger} B_1 \tilde{D}_{\perp}^T \\ \hline 0 & -(B_1 + HD_{21})^T & D_{21}^T & \tilde{D}_{\perp}^T \\ F & 0 & 0 & 0 \end{array}\right]$$

conjugating by $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$,

$$\begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix} \begin{bmatrix} \tilde{N}_{21}^{*} & \tilde{N}_{\perp}^{*} \end{bmatrix} = \left[\begin{array}{cc|cc} A + HC_2 & 0 & 0 & 0 \\ 0 & -(A + HC_2)^T & C_2^T & -Y^{\dagger} B_1 \tilde{D}_{\perp}^T \\ \hline 0 & -(B_1 + HD_{21})^T & D_{21}^T & \tilde{D}_{\perp}^T \\ F & FY & 0 & 0 \end{array}\right]$$

which verifies the claim.

Q.E.D.

Putting these results together yields

$$\begin{bmatrix} N_{12}^* \\ N_\perp^* \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \end{bmatrix} \begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix} \begin{bmatrix} \tilde{N}_{21}^* & \tilde{N}_\perp^* \end{bmatrix} \tag{16}$$

$$= \left[\begin{array}{c|cc} -(A+B_2F)^T & (C_1+D_{12}F)^T D_{11} + XB_1 & 0 \\ \hline B_2^T & D_{12}^T D_{11} & I \\ D_\perp^T C_1 X^\dagger & D_\perp^T D_{11} & 0 \end{array}\right] \left[\begin{array}{c|cc} -(A+HC_2)^T & -C_2^T & Y^\dagger B_1 \tilde{D}_\perp^T \\ \hline (B_1+HD_{21})^T & D_{21}^T & \tilde{D}_\perp^T \\ -FY & 0 & 0 \end{array}\right]$$

Note that this is the cascade of two systems with all of their poles in $\mathbb{C}_+$. Thus, projection onto $H_2 + \mathbb{C}$ leaves only the constant term. Therefore, in the L_2 case :

Theorem :

$$Q_{opt} = D_{12}^T D_{11} D_{21}^T \tag{17}$$

The $\alpha=\infty$ case is somewhat more complicated but can quickly be reduced to a standard matrix best approximation problem. Suppose that from (8) we want to find all $Q \in RH_\infty$ such that

$$= \left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\alpha \leq \gamma. \tag{18}$$

From Chapter 2.1 we know that (18) may be reduced without loss of generality to

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty \leq \gamma \quad \text{for} \quad \gamma > \hat{\gamma} = \left\| R_{21} \right\|_\infty \tag{19}$$

Recall that

$$\left\| \begin{bmatrix} R_{11}+Q \\ R_{21} \end{bmatrix} \right\|_\infty \leq \gamma \quad \text{for} \quad \gamma > \hat{\gamma} \tag{20}$$

iff

$$\left\|(R_{11}+Q)(\gamma^2 I-R_{21}^{*}R_{21})^{-\frac{1}{2}}\right\|_{\infty} \leq 1 \qquad (21)$$

iff

$$\left\|G + \hat{Q}\right\|_{\infty} \leq 1 \qquad (22)$$

where $G=R_{11}M^{-1} \in RH_{\infty}$ and $\hat{Q}=QM^{-1}$. Solving (22) for $\hat{Q} \in RH_{\infty}$ solves (20) for $Q \in RH_{\infty}$. Note that $Q=\hat{Q}M$ is in RH_{∞} if $\hat{Q}$ is, since $M \in RH_{\infty}$ by construction.

The final step in the rational case then involves solving (22) for $\hat{Q} \in RH_{\infty}$. This is a standard mathematical problem of approximating an L_{∞} matrix by an H_{∞} matrix.

Part 3. Large Space Structure Control

1. LSS Control Perspective

2. Modeling and Analysis Issues

3.1 LSS Control Perspective

This part will outline the main issues involved in applying the methods of the previous parts to large space structure (LSS) control problems. The most fundemental issue is how the performance specifications and uncertainty can be put into the general framework developed in Part 1. For the most part, LSS control problems involve the usual problems of providing tracking and disturbance rejection in the presence of plant uncertainty and noise.

A significant distinguishing feature of LSS control seems to be that there is a great deal of structure associated with the plant uncertainty. On the one hand, this suggests that the methods of the previous sections, which focus attention on structured uncertainty, are uniquely suited to handle LSS control problems. On the other hand, it will require significant application-specific modeling and analysis to exploit the known structural properties of LSS uncertainty.

The remainder of this section will outline the main issues in control of LSS's, with emphasis on those associated with large antennas. The discussion will be kept as elementary as possible, and will avoid the use of the more advanced of the mathematical techniques developed in the previous parts of these notes. With this as a foundation, the next section will discuss specific modeling issues and how they relate to the methods of the rest of these notes.

From a controls perspective, the problem is to achieve the specified mission performance from an incompletely known spacecraft in the face of uncertain disturbances. Such a problem invariably requires the use of feedback. Because the effect of feedback is critical to spacecraft performance a

review of the feedback fundamentals relevant to this problem is in order. By examining the properties of feedback in a general setting, without regard to the technique used to generate the feedback law, fundamental relationships between performance, robustness and feedback control loop properties become evident. This will make it easier to appreciate the role that the specific methods developed in the preceding sections can play in analysis and synthesis of controllers for large space structures.

Feedback Improves Performance

A typical feedback situation for a space pointing mission is shown in Fig. 3-1. Here we have disturbances acting on a structure to upset the primary goal of the mission, namely to keep line-of-sight[1] (LOS) errors smaller than a specified level, ε,

$$\left| \mathrm{LOS} - \mathrm{LOS}_c \right| \leq \varepsilon .$$

It is assumed that measurements of LOS are imperfect due to sensor dynamics, sensor noise and (possibly) the need to infer LOS from related measurements. In typical feedback fashion, the sensed response is compared to the desired or commanded value and the error used by a controller to generate a control signal to drive the actuators on the structure.

Each of the elements of this loop can be represented by its transfer function (TF) relating the Laplace transform of the response to the Laplace transform of the forcing function. The error response of the structure to commands, LOS_C disturbances, D, and sensor noise, N, can then be developed by standard feedback equation manipulation. Letting

[1]For simplicity of exposition in this section, we use the term line-of-sight (LOS) generically to refer to all controlled variables of interest — LOS, wavefront, etc.

$D_o = G_D D =$ disturbance as seen at the output LOS in the absence of feedback

$L = KGT =$ feedback loop transfer function (i.e., transmission around the feedback loop)

and $\Delta T = T_{true} - T =$ LOS sensor uncertainty

we get

$$(\mathrm{LOS} - \mathrm{LOS}_c) = \left[\frac{1}{1+L}\right](D_o - \mathrm{LOS}_c) - \left[\frac{L}{1+L}\right]\frac{(N + \Delta T \mathrm{LOS}_c)}{T}$$

This equation relates system performance to each error source. Its various terms can be interpreted in many ways, such as amplitudes of sine waves or as signals in L_2. In either case, four immediate consequences can be seen:

Consequence No. 1 –The loop transfer function must be large to achieve small errors. This follows from the first term on the right hand side of the error equation. In fact, in order to meet our specified error level, we must have

$$\varepsilon \left| 1+L \right| > \left| D_o - \mathrm{LOS}_c \right|$$

Thus, at those frequencies where either the disturbance responses or the commands are large compared with ε, we require $|L| \gg 1$.

Consequence No. 2 –Sensor noise must be small enough. This follows from the second term, which requires that

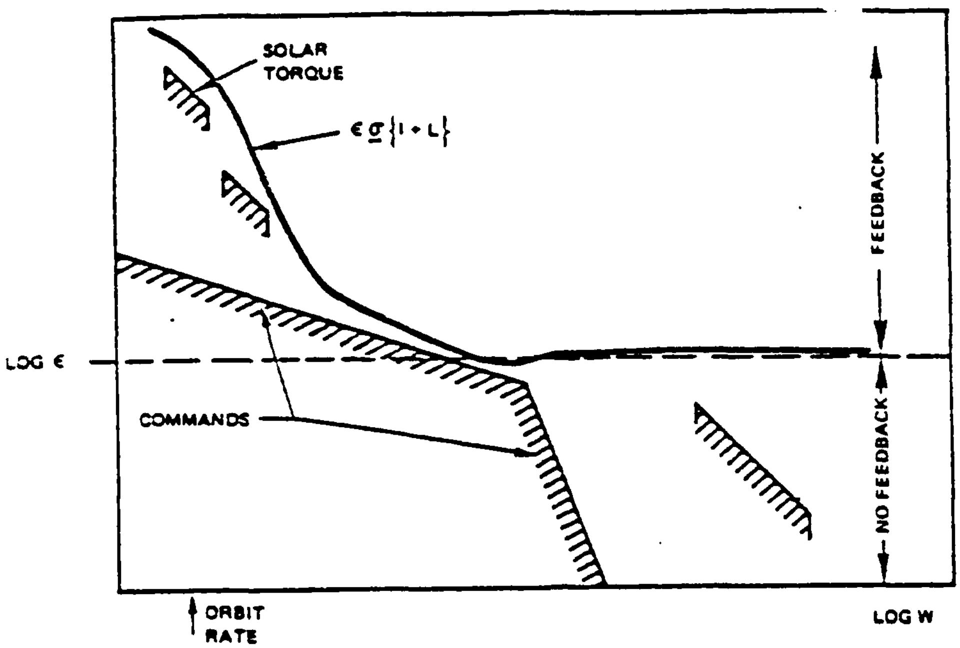

Figure 3-3. Suitable Control Solution

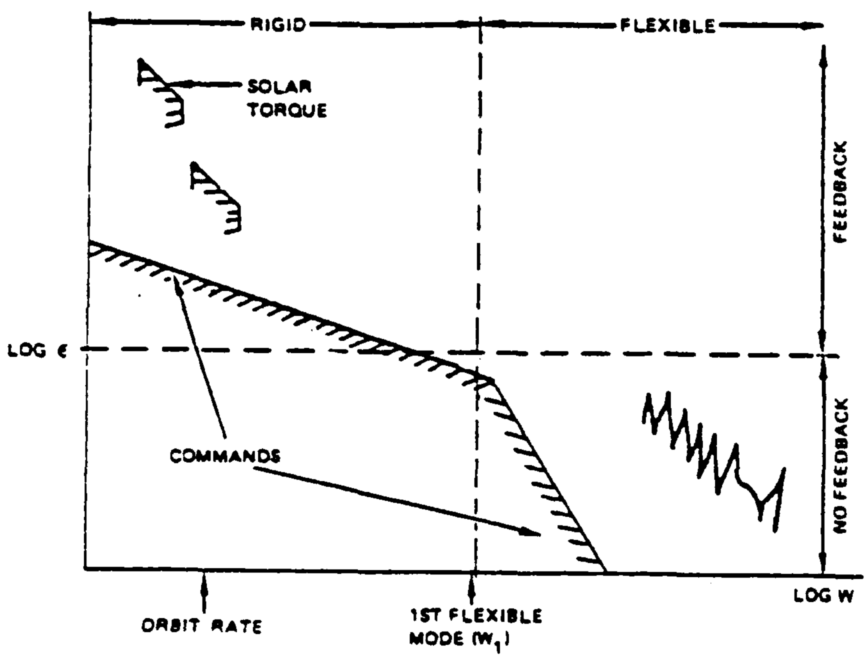

Figure 3-4. Effect of Spacecraft Flexibility on LOS Control

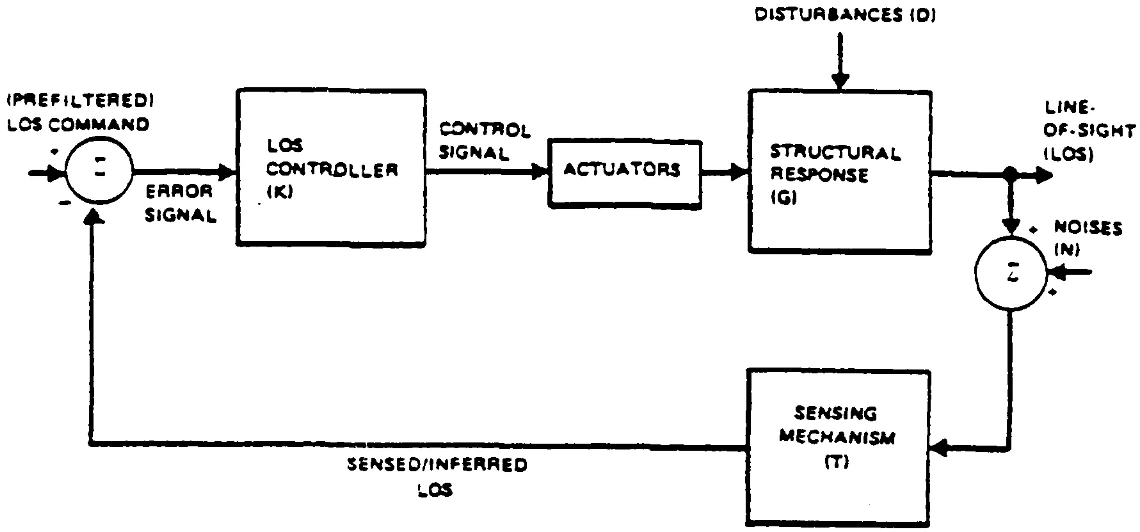

Figure 3-1. LOS Feedback Loop

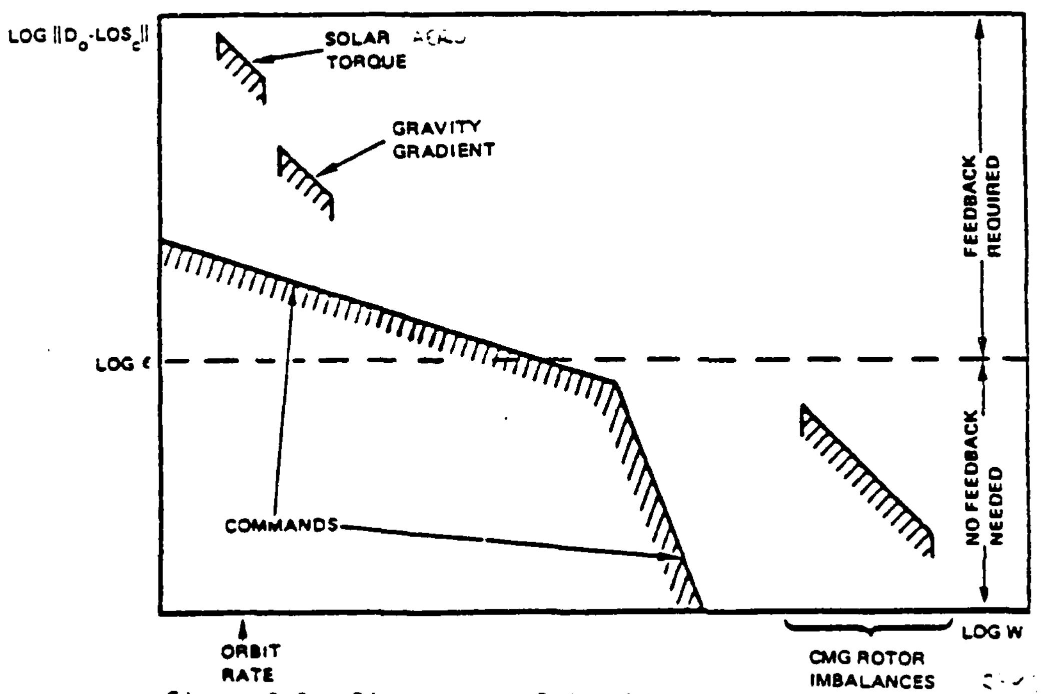

Figure 3-2. Disturbances Drive Control Requirements

seem to be two obvious alternatives to this direct approach. The most desirable method would be to actually check for the robustness with respect to the real parameter variations along with those modelled as complex. This clearly involves a generalization of the existing SSV and no reliable method exists for computing such an object in any but the most simple cases.

A more promising alternative, at least in the near term, seems to be to replace the *effect* of the real parameter variation with a complex one. That is, instead of using the same linear fractional transformation on the Δ and simply replacing the real parts of Δ with complex perturbations, the linear fractional transformation is changed as well so that the *set* of plants described by the new representation closely matches the original set of plants. Preliminary investigations of this approach appear promising, but no systematic procedure has been developed for achieving this model simplification.

Both of these approaches to the real parameter robustness problem have been investigated and will continue to be important research directions. The results available to date are rather fragmented and incomplete. While we have had remarkable success in applying the SSV to several example problems (including the space shuttle) involving real parameter variations, the details of the techniques were somewhat problem-specific. It is expected that a more coherent view of the real perturbation problem will emerge in the next year.

noise model, can be rearranged to fit the general framework of Part I, shown in Figure 13. The major sources of uncertainty in this model are summarized in Tables 1,2 and 3. In these tables "external" and "internal" are used to describe disturbances that are generated physically either outside or inside, respectively, of the spacecraft. In the sense of Part I, all these disturbances are external.

It is clear that the model depicted in Fig. 13 will have a great deal of structure; the Δ will have many blocks. Thus, the methods of Part I using the SSV are clearly going to play a critical role in the analysis and synthesis of controllers for LSS's. It is unlikely that it will be possible to reliably handle the inherent uncertainties of LSS control using the paradigm of either stochastic optimal control or standard robust multivariable control theory based on singular values.

On the other hand, the application of SSV techniques to LSS control appears to be a challenging problem. The large number of blocks in a reasonable perturbation model of a LSS will stress the current experimental software, so improvements in computation and approximation methods would be desirable. Another critical issue is that a naive application of the SSV to Fig. 13 for large space structures could result in very pessimistic results. The reason for this is that the natural way to represent uncertain structural mass, stiffness, and damping characteristics is with *real* parameter variations. Thus, many of the blocks in Fig. 13 would be real, and application of the SSV would actually involve checking robustness characteristics with respect to complex variations.

Simply replacing real perturbations with complex ones of the same magnitude would typically be very conservative in LSS control problems. There

"pseudo" truth model, which we recommend, is to introduce perturbations of appropriate magnitude in all uncertain parameters of the evaluation model.

Disturbance / Command Models—The most critical disturbances for controlling large space antennas are likely to be high-frequency vibrations due to CMGs and other rotating machinery, plus distributed forces/torques due to this same equipment, coolant turbulence, etc. These may be approximated as (1) discrete-frequency components, provided a sufficient number of harmonics are included and uncertainty in each of the frequencies is accounted for and (2) broadband disturbances. These latter disturbances will likely be dominated by the large command requirements for slew maneuvers. Bandwidth of these command requirements will be minimized by the use of appropriate command shaping where applicable. Although the numbers will change, the general characteristics of these disturbances is illustrated in the plot of Fig. 3-10.

Sensor / Actuator Models--Linear models for sensors and actuators will include all relevant dynamics for frequencies out to roughly a decade beyond the desired control loop gain crossover. The effects of unmodeled high-frequency dynamics and any significant nonlinearities can be approximated using the frequency-dependent multiplicative uncertainties illustrated in Fig. 3-11 and 3-12.

Analysis Issues

Figures 3-9, 3-11, and 3-12 can be combined to obtain the desired nominal plus perturbation model of the LSS that is required to use the analysis tools of Part I. This perturbation model, together with a disturbance and

$$G(j\omega) \triangleq \sum_{i=1}^{n} g_i(s) c_i b_i^T$$

where

$$g_i(s) \triangleq \frac{1}{s^2+2\zeta_i\omega_i s+\omega_i^2}$$

This model, as usual, admits a state-space representation of the form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where

$$A = \text{diag}\left\{\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix}\right\}$$

$$B = \text{col}\left\{\begin{bmatrix} 0 \\ b_i^T \end{bmatrix}\right\}$$

$$C = \text{row}\left\{\begin{bmatrix} c_i & 0 \end{bmatrix}\right\}$$

$$D = 0$$

Because the truth model is not available to the designer, control designs developed using the design model can only be directly verified using the evaluation model, which accounts only for the known (but largely unstructured) errors, $\Delta G_e(j\omega)$. Thus he must "design" to accommodate both these known errors as well as unknown errors, which include both structured errors (ΔA, ΔB, ΔC) and largely unstructured errors, $\Delta G_t(j\omega)$. It is, of course, possible to define a "pseudo" truth model by deliberately introducing perturbations in the structural data used to define the finite-element evaluation model, as was done for some of the ACOSS studies. This approach, unfortunately, is prone to making the same approximation errors in the "pseudo" truth model as exist in the evaluation model. A safer approach to defining a

finite-element modeling technique (e.g., NASTRAN). It has the form

$$G_e(s) \triangleq \sum_{i=1}^{n_e} g_{ei}(s) c_{ei} b_{ei}^T$$

where

$$g_{ei}(s) \triangleq \frac{1}{s^2+2\zeta_{ei}\omega_{ei}s+\omega_{ei}^2} \quad i=1,2,\ldots,n_e$$

Note that this model differs from the truth model in two respects. First, it is of lower (finite) order. This difference can generally be approximated as a bounded frequency-dependent multiplicative error, ΔG_t, which is reflected either to the input or output (as illustrated in Fig. 3-9) of the system. The second difference is a more systematic one – the model parameters are different. This is due to a number of factors – poorly known structural properties, too coarse a finite-element grid, nonlinearities, etc. Such errors are highly structured and generally cannot be reflected to either the input or output of the system, without substantially increasing the corresponding bounds for these errors.

The third model, or design model, is a lower-order approximation to the evaluation model, which is used for control design. It is derived from the evaluation model by means to be described shortly. Since it is used for control design, it should not differ greatly from the evaluation model, except at high frequencies well beyond desired control loop gain crossover. These differences can generally be represented by the bounded frequency-dependent multiplicative error, ΔG_e, illustrated in Fig. 3-9. Depending on control requirements, this model may sometimes differ also at low frequency. The design model has the general form

3.2 Modeling and Analysis Issues

We next examine models and model uncertainty for critical elements of the control design problem, and to see how the LSS control problem fits into the framework of Part I.

Model Definition

Spacecraft Models--To properly motivate the issue of model error, we define three spacecraft models, which are illustrated in Fig. 3-9. The first model, or truth model, is supplied by "Mother Nature." It is an infinite-dimensional model which can be represented in the form (assuming either position or attitude outputs ond force or torque inputs)

$$G_T(s) \triangleq \sum_{i=1}^{\infty} g_{Ti}(s) c_{Ti} b_{Ti}^T$$

where

$$g_{Ti}(s) \triangleq \frac{1}{s^2+2\zeta_{Ti}\omega_{Ti}s+\omega_{Ti}^2} \quad i=1,2,....$$

Here ω_{Ti} and ζ_{Ti} denote frequency and damping for the ith mode while b_{Ti} and c_{Ti} denote input and output influence coefficient vectors for all relevant input and output nodes. Line-of-sight and wavefront errors can also be expressed in this same form by appropriate definition of the c_{Ti}'s, as can disturbance inputs by appropriate definition of the b_{Ti}'s. Rate and acceleration outputs can be handled by introducing s and s^2 numerator terms to the g_{Ti}'s.

The second model, or evaluation model, is a high-order finite-dimensional approximation to the truth model, which is typically derived from some

Note that the plot in Fig. 3-7 provides explicit requirements for vibration control. For example, the important controlled modes are identified as those above the ε-line. Also, the amount of damping each controlled mode requires is clearly shown. Finally, the effect and adequacy of any structural solutions, such as stiffening, can be clearly assessed.

Class IV: Integrated Pointing and *Vibration Control*—As the ε-line moves still lower, it eventually lies below the average response of the flexible structure. Damping augmentation alone is no longer sufficient. In fact, the LOS pointing controller must now of necessity provide active feedback control in the flexible frequency region as shown in Fig. 3-8. The significant feature of this problem is that the LOS itself must now be measured accurately (or inferred from related measurements) up to very high frequency.

Applicability to Large Space Antennas

In view of the inevitable overlap between structural mode frequencies and control bandwidth required to meet line-of-sight and antenna surface shape control requirements in the face of high frequency vibration disturbances and high-bandwidth slewing command requirements, the large space antenna control problem falls into Class IV. Thus we now examine modeling and control design issues that are relevant to this class of problems.

increase, we find that some high-frequency modes exceed the ε-line. This is illustrated in Fig. 3-6. If high frequency disturbances are generated internally, however, it is often possible to isolate them (e.g., shock-mounted CMGs). This decreases $\left\|D_s\right\|$ and reduces the problem to Class I, for which solutions are known.

Class III: *Attitude Control with Isolation* and *Vibration Control*— There are limits, however, to how much isolation can be provided. One of the major limitations, for example, occurs when the LOS controller must act through the isolator to point the payload (as was true for the ACOSS II model). Rigid body stability requirements then impose a lower limit on isolation frequency. For example, assume we can tolerate no more than 45 degrees of phase shift from the isolator at the LOS controller crossover frequency, ω_c. Furthermore, assume we wish to use a passive second-order isolator. Then the Bode gain/phase relations require that

$$\omega_{iso} \geq 12.5\omega_c$$

where ω_{iso} is the lowest permissible frequency of the isolator. Similar calculations can be made for other situations such as active isolation or higher-order passive isolation.

Thus, as disturbances increase or specifications get tighter, we must eventually face feedback control of flexible modes. This is illustrated in Fig. 3-7. If only resonant peaks lie above the ε-line, it is sufficient to smooth the disturbance response to more closely follow the average, shown dotted in Fig. 3-7 to return to the conventional problem. This is exactly what vibration damping does.

Note that 1 and 2 together allow for a-priori assessment of component bandwidths in terms of mission specs (ε) and disturbance environments ($\left\|D_o - LOS_c\right\|$) alone.

The concepts discussed above apply to flexible spacecraft as well as rigid. We merely include the effects of flexibility in calculating the open loop disturbance responses. Figure 3-4 is an example. Again, the ε-line is introduced to determine the frequencies where feedback control is necessary. Now, however, we also introduce a second line, a vertical line at $\omega = \omega_1$, which serves to partition the frequency range into rigid and flexible. Together these two lines serve to categorize the control problem.

Control Classes

The control problem can be categorized according to where the ε-line and the first flexible mode flex-line occur relative to the disturbance responses. These categories describe classes of pointing control problems with different degrees-of-difficulty and different solution approaches. Note tht these classes are unrelated to those of Part 1.

> *Class I*: *Conventional Attitude Control*—When all disturbance limits to the right of the flex-line remain below the ε-line (as illustrated in Fig. 3-5), no feedback control of the flexible modes is required. We then have a conventional rigid-body design problem which is routinely solved on current generation spacecraft.
>
> *Class II*: *Attitude Control with Isolation*—If either the LOS specification gets tighter (ε-line decreases) or the disturbances

is best understood graphically by plotting the right hand side, $\left\| D_o - LOS_c \right\|$, of Equation (1). Such a plot is shown for a rigid spacecraft and for typical disturbances in Fig. 3-2 on a log-log scale. For example, solar torques cause open-loop LOS deflections near the orbital frequency. CMG rotor imbalances cause LOS errors at higher frequencies. All such disturbances can be pulled together on the same plot according to their effect on the LOS output (i.e., $\left\| D_o \right\|$). In addition, the command spectra can be included as the transform of the desired response to LOS commands. Together these terms give $\left\| D_o - LOS_c \right\|$.

Superimposed on Figure 3-2 is an ε-line which defines the allowable error. Consequence No. 1 can be used to determine the requirements on L. In particular, for those parts of the figure which lie <u>below</u> the ε-line, no feedback is required, i.e., $\{ \bar{\sigma} \{ L \} << 1 \}$ at those frequencies. However, for those parts which lie above the line, we require $\underline{\sigma} \{ L \} > 1$ in order to ensure reduction of errors to the ε-level in the closed loop.

These observations suggest that Fig. 3-2 can be used to graphically determine the suitability of any L, and thus of any compensator K, by directly sketching $\varepsilon \underline{\sigma} \{ 1+L \}$. Such a plot is shown as Fig. 3-3. We observe that:

1. It is desirable to have the $\varepsilon \underline{\sigma} \{ 1+L \}$ line follow as closely as possible to the disturbance limits. This minimizes control authority.

2. The point where $\varepsilon \underline{\sigma} \{ 1+L \} = \varepsilon$ is the crossover frequency, ω_c. All hardware components involved in this loop must have bandwidths higher than this frequency.

In the minds of most engineers, these fundamental consequences of feedback are associated with the classical single-input single-output feedback theory. It is clear from Part 1 of these notes that they are equally true for more complex multi-input multi-output situations. In fact, this has been made clear by much earlier research at Honeywell on extensions of classical concepts to multivariable problems using singular values. This corresponds to the case 2 framework described in Part 1. Using this framework , Consequences No. 1 through No. 4 can be restated for the multivariable case as follows:

1. $\varepsilon \underline{\sigma}\left\{I + L\right\} > \left|\left| D_e - LOS_c \right|\right|$
2. $\left|\left| T^{-1}N \right|\right| < \varepsilon$
3. $\bar{\sigma}\left\{T^{-1}\Delta T\right\} < \dfrac{\varepsilon}{\left|\left| LOS_c \right|\right|}$
4. $\underline{\sigma}\left\{I + L\right\} > \bar{\sigma}\left\{\Delta L\right\}$

Here $\bar{\sigma}\{\bullet\}$ and $\underline{\sigma}\{\bullet\}$ are the maximum and minimum singular values of the indicated matrices and $\left|\left|\bullet\right|\right|$ represents the magnitude (norm) of the indicated vector. $\bar{\sigma}\{\bullet\}$ can be interpreted as the maximum gain which the matrix can produce (at a frequency) and $\underline{\sigma}$ can similarly be interpreted as the minimum gain. This point of view is discussed in more detail in [DSt2].

Graphical Interpretation

There is no need to delve into the intricacies of matrix theory in order to understand the *fundamentals* of feedback. Consequence No. 1, for example,

$$\left|\frac{N}{T}\right| < \varepsilon \quad \text{whenever} \quad |L| \gg 1$$

Consequence No. 3 --Sensor uncertainties (errors in T) must be small enough. This also follows from the second term, which requires that

$$\left|\frac{\Delta T}{T}\right| < \frac{\varepsilon}{|\mathrm{LOS}_e|} \text{ whenever } |L| \gg 1$$

This result specifies the uncertainty which can be tolerated in the LOS derivation at each frequency. In essence, it establishes requirements for alignment and calibration of the LOS reference system.

Consequence No. 4 --Model uncertainties (errors in L) must be small enough. This follows from both terms. Letting errors in L be ΔL, these terms require that

$$1 + L + \Delta L \neq 0$$

for all frequencies and all ΔL. One way to ensure that this is true is to require

$$|\Delta L| < |1 + L|$$

for all frequencies and all ΔL.

This is a statement of the robustness requirement of feedback systems and quantifies the amount of uncertainty which can be tolerated without loss of stability.

Multi-Inputs and Multi-Outputs

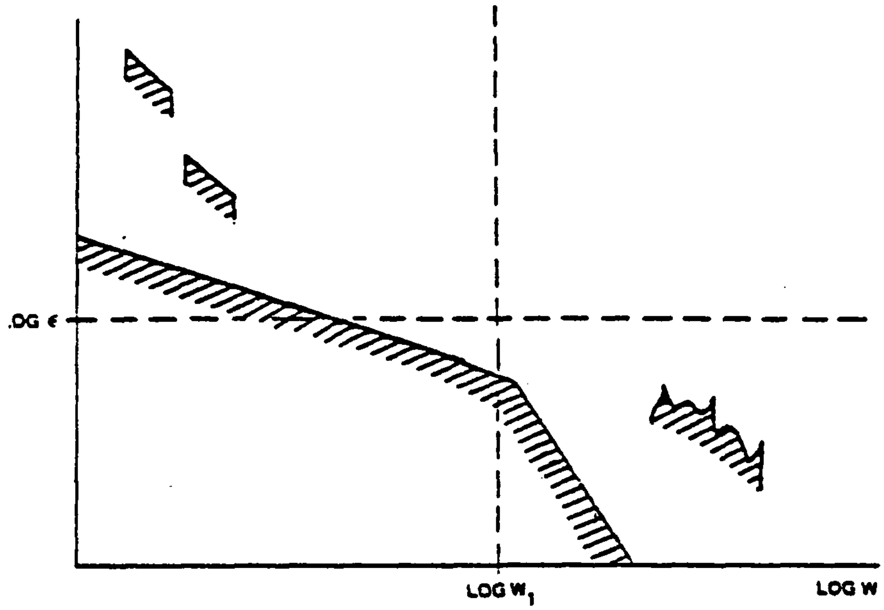

Figure 3-5. Class I: Conventional Attitude Control

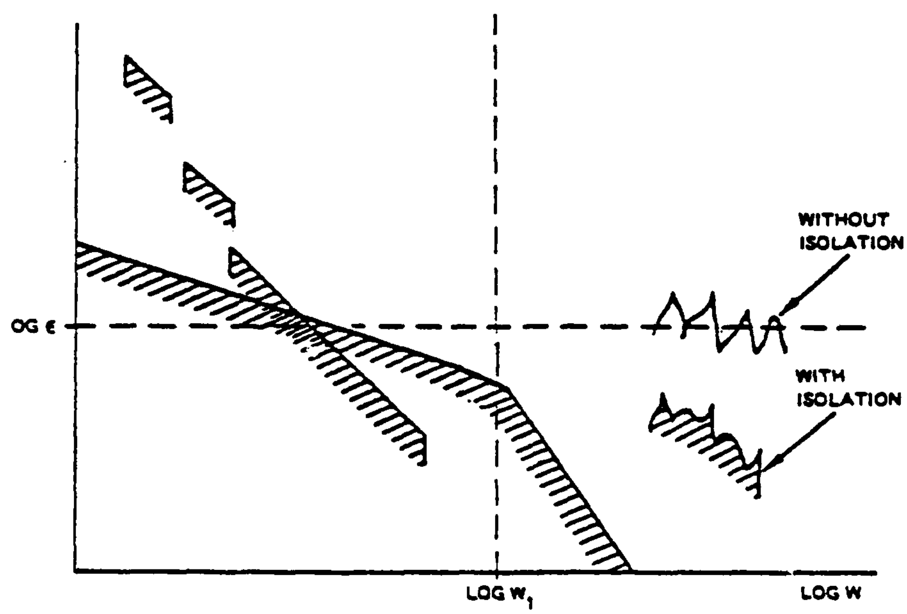

Figure 3-6. Class II: Attitude Control with Isolation

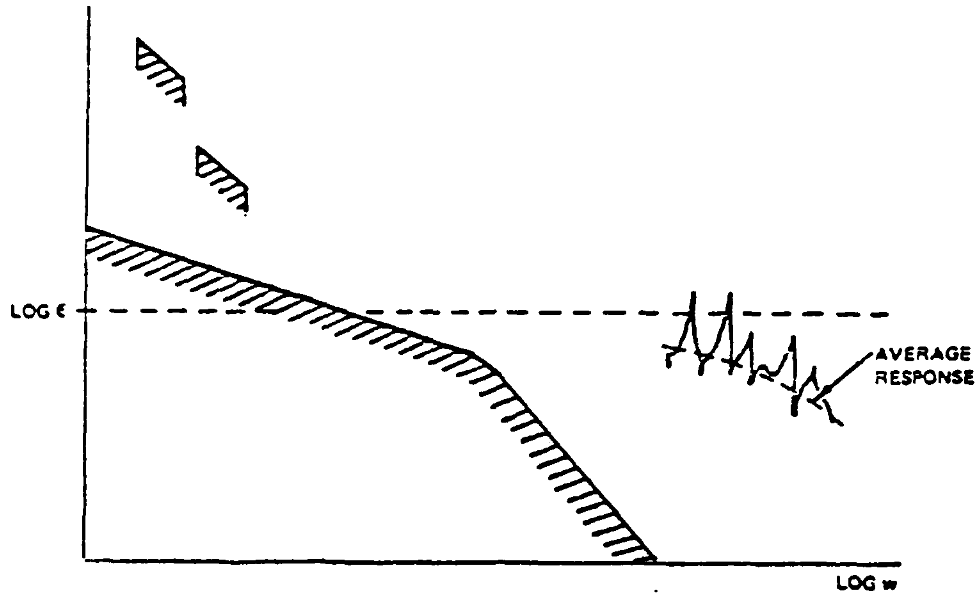

Figure 3-7. Class III: Attitude Control with Isolation and Vibration Control

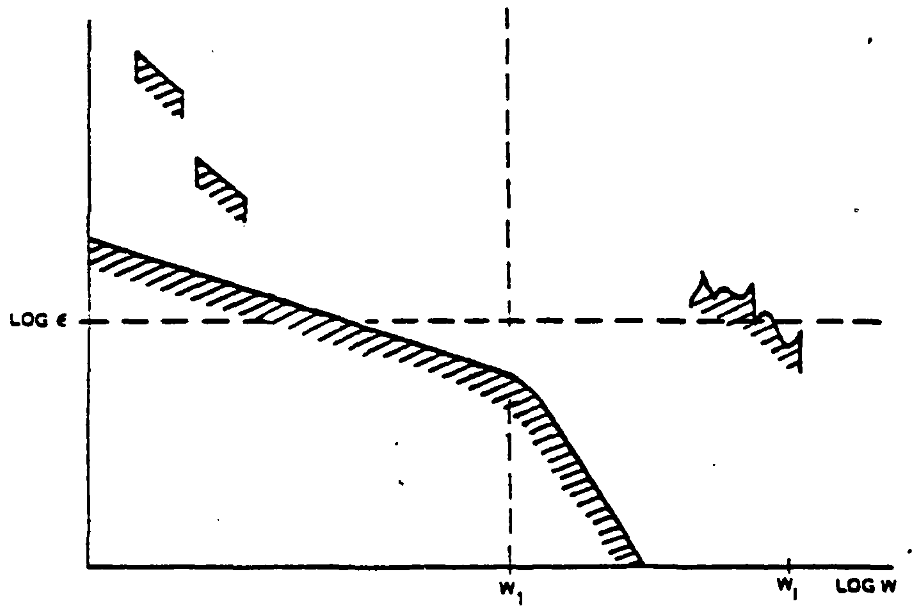

Figure 3-8. Class IV: Integrated Pointing and Vibration Control

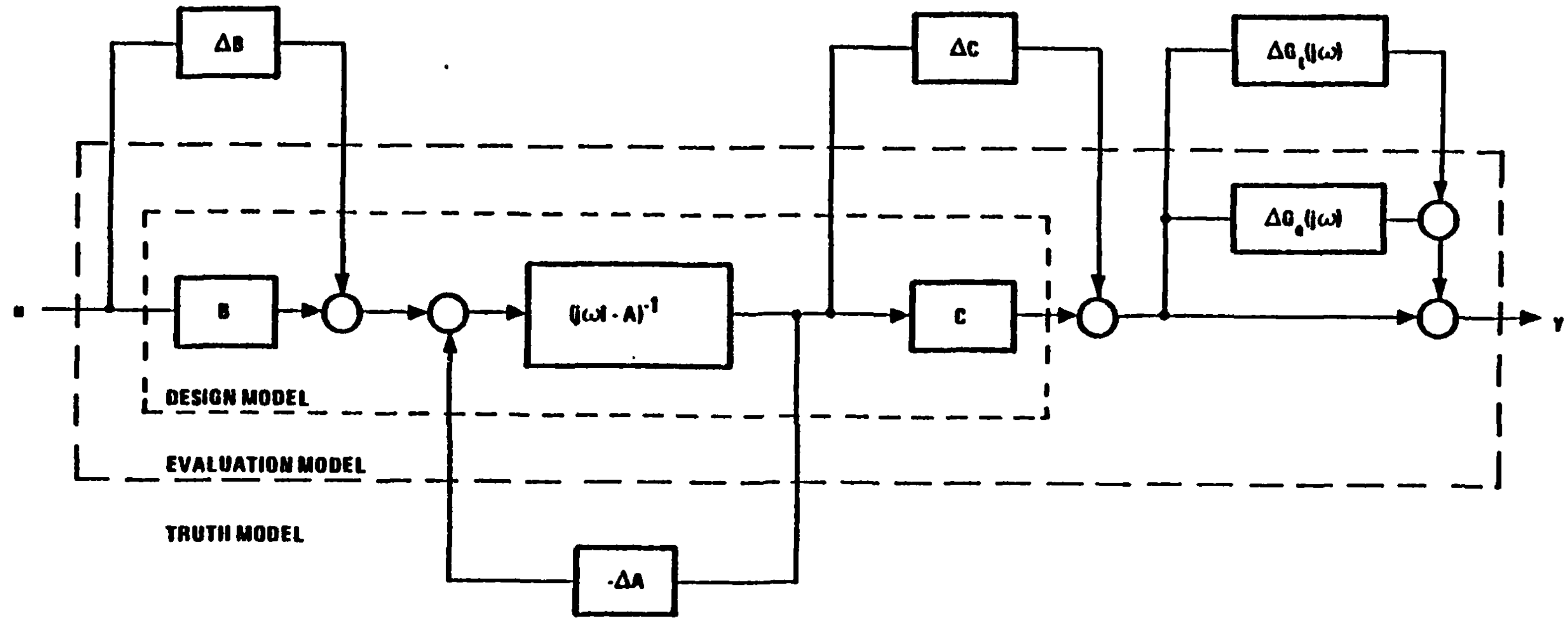

Figure 3-9. Spacecraft Model with Structured and Unstructured Uncertainty

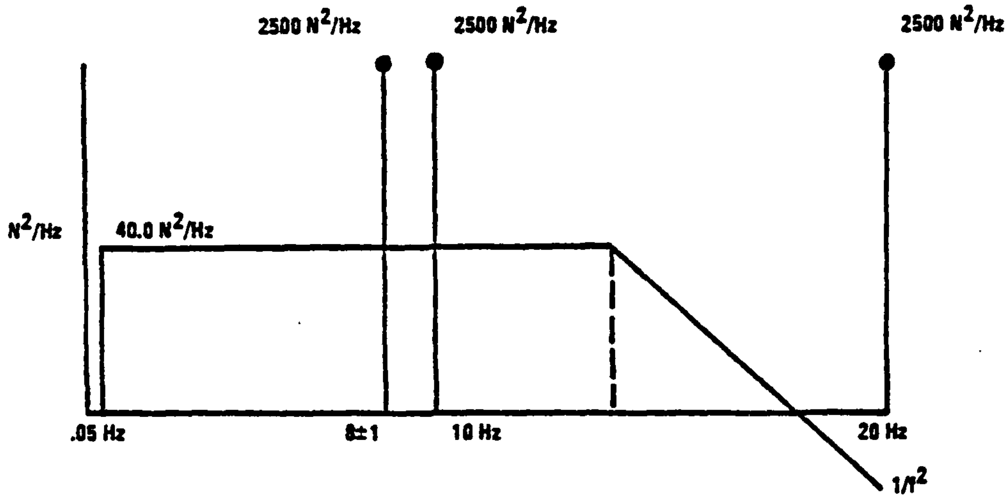

Figure 3-10. Disturbance/Command PSDs (VCOSS Studies)

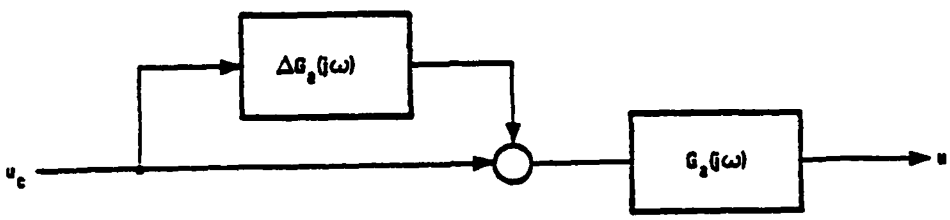

Figure 3-11. Actuator Model with Unstructured Uncertainty

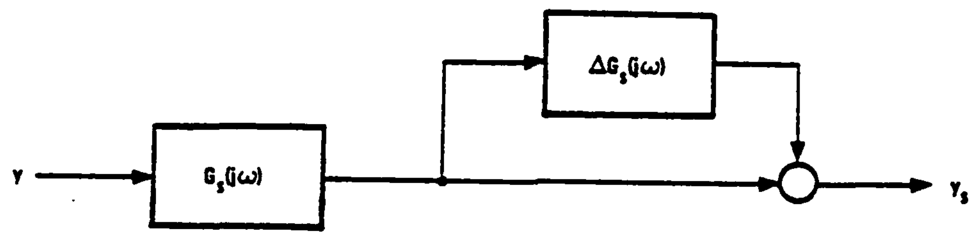

Figure 3-12. Sensor Model with Unstructured Uncertainty

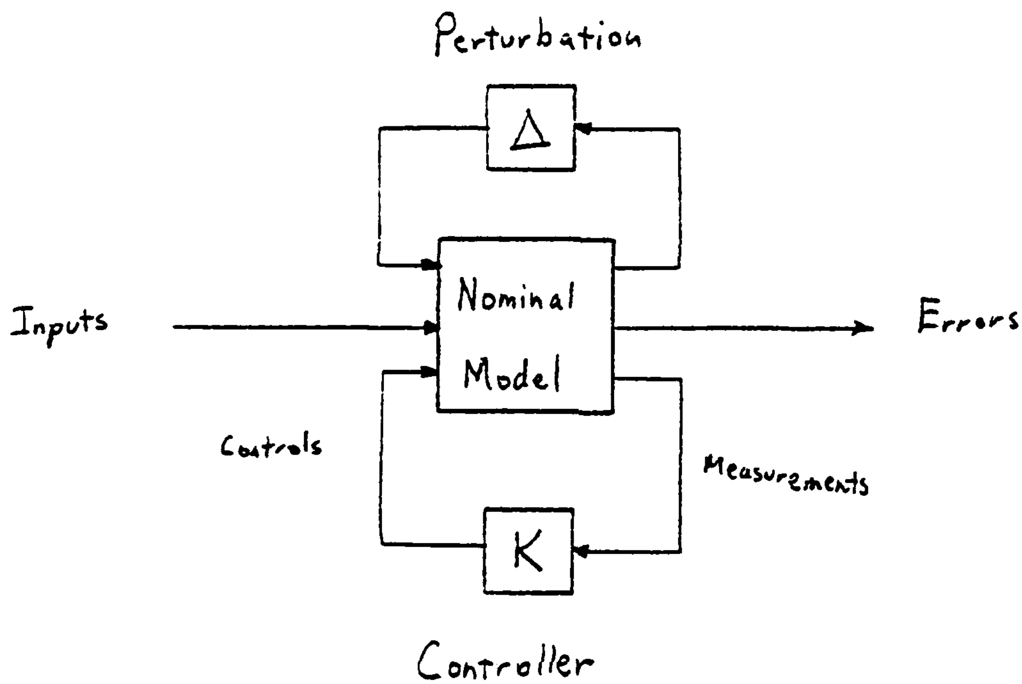

Figure 13. General Structure

Table 1. LSS Uncertainty: *External* Disturbances and Noises

Source	Frequency Range	Comments
Aerodynamic	Low (Orbit Rate)	Significant for low-altitude orbits, $h < 500\ km$ (Dominant "external disturbance for $h < 250\ km$)
Solar	Low (Orbit Rate)	Independent of altitude (Dominant "external" disturbance at geosynchronous altitude)
Gravity Gradient	Low (Orbit Rate)	Most significant at low altitude, but relatively weak for $h < R_e$
Earth Magnetic	Low (Orbit Rate)	Most significant at low altitude but relatively weak function of altitude
Thermal	Low (Orbit Rate)	Independent of altitude

Table 2. LSS Uncertainty: *Internal* Disturbances and Noise

Source	Frequency Range	Comments
Stationkeeping Residual Jet Imbalances	Mid. (Steps)	Typically the dominant "internal" disturbance
Solar Panel Drive	Mid. (Step)	
Sensor/Actuator Errors	Low-Mid	Attitude Reference System Biases/Alignment/Noise
	Mid-High	Rate Gyro Noise/Drift
	All	Reaction Wheel/Control Moment Gyro
Vibration	Mid-High Narrow Band	Reaction Wheels / Control Moment Gyros (small) Cryogenic Coolers, High-energy Weapons } large when present

Table 3. LSS Uncertainty: Perturbations

Source	Frequency Range	Comments
Actuators/Sensors	High	Unknown or approximated high-frequency dynamics. Often modelled as $I + \Delta$
Structure	All	Uncertain Mass, Stiffness, Damping. Two modelling options: 1) Directly as parameter variations 2) Indirectly as uncertain modal frequency, damping, influence coefficients.
Truncation	High	Neglected dynamics, many modelling options, primarily unstructured

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